The Generous Donor

Suppose you are the director of a charity organization. A very wealthy supporter of your organization has passed away with no heirs. This wealthy supporter has indicated that she wants all her money to go to your charity. Her accountants have given you a choice between two payment plans. Your job to choose between these plans:

First Plan

$100,000,000 the first week,

4/5 of the previous week's amount for every week beyond ...

The payments are to continue forever.

Second Plan

$8,000,000 the first week,

$8,000,000/2 (= $4,000,000) the second week,

$8,000,000/3 (= $2,666,666.6666) the third week, and so on...

The payments are to continue forever.

Which payment plan will provide your organization with the greater contribution in the long run? Why?

To answer these questions, we first need to examine the total contribution each payment plan would make as of the nth week. Then we need to examine these values in the long run – that is, as $n \to \infty$.

Let's begin by introducing variables and restating the problem in mathematical terms.

Let F_n denote the total contribution made to your organization after n months if the *first payment plan* is chosen.

Let S_n denote the total contribution made to your organization after n months if the *second payment plan* is chosen.

1. Consider the first payment plan. Explain how F_n can be evaluated using F_{n-1}. Therefore write an equation evaluating F_n in terms of $F_0 = 100,000,000$.

2. Your formula for F_n should look like $F_n = ar^n$ where r is a constant and a is the initial amount. $\sum_{n=0}^{\infty} ar^n$ is called a **geometric series**. We can show that the value of the infinite sum is given by the formula $\frac{a}{1-r}$.

3. How much money would your charity eventually (i.e. as $n \to \infty$) receive under this first plan?

4. Now consider the second payment plan. Explain why S_n CAN NOT be evaluated as a geometric sum.

How can we determine how much money would eventually be received from the second plan? Use the Integral Test!

The Integral Test for Infinite Series

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with all positive terms and let $f(x)$ be a monotone decreasing function on $[1, \infty)$ satisfying $f(k) = a_k$ for every positive integer k. Then

$$\int_1^{\infty} f(x)\,dx \text{ converges if and only if } \sum_{k=1}^{\infty} a_k \text{ converges.}$$

In other words, the infinite series and improper integral converge or diverge together!

The above statement may be proved *by picture*. What is the appropriate picture? Begin by graphing a monotone decreasing function and then draw rectangles of width 1 as appropriate. The area of the rectangles should be greater than the area under the curve, and this proves that if the infinite series converges, then the improper integral converges. The other part of the theorem may be proven by shifting all the rectangles in your picture one unit to the left. *Explain.*

We now apply this to determine the convergence or divergence of a variety of infinite series. In each case, given an infinite series, we need to introduce a function $f(x)$. Do this by viewing the terms of the series as the output values of a function (think of a_k as $a(k)$).

Do the following infinite series converge or diverge? Along with the integral test, use the ideas of comparison and eventual comparison as necessary.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{21}{4k^4 + 17}$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{e^{k^3}}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^k}$