Inverse Functions

Inverses and Identities

Many operations on sets of numbers or functions have an *identity element* and *inverses* in the set. Important examples include addition, multiplication, and composition of functions.

Addition

There is exactly one real number a with the property that

$$a + x = x + a = x, \quad \text{for all } x \in \mathbf{R}.$$

This number, the *additive identity*, is $a = 0$.

If b is a real number, there is exactly one real number c such that

$$b + c = c + b = 0.$$

This number, the *additive inverse* of b, is $c = -b$.

We can extend these ideas from numbers to functions. There is exactly one function $h : \mathbf{R} \to \mathbf{R}$ such that

$$h(x) + f(x) = f(x) + h(x) = f(x), \quad \text{for all } f : \mathbf{R} \to \mathbf{R}.$$

This function, the *additive identity* for functions, has the formula $h(x) = 0$, for all $x \in \mathbf{R}$.

The *additive inverse* of the function f is the function g such that

$$f(x) + g(x) = g(x) + f(x) = h(x) = 0.$$

In fact, $g(x) = -f(x)$. The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ across the x-axis.

Multiplication

There is exactly one real number a with the property that

$$a \cdot x = x \cdot a = x, \quad \text{for all } x \in \mathbf{R}.$$

This number, the *multiplicative identity*, is $a = 1$.

If $b \neq 0$ is a real number , there is exactly one real number c such that

$$b \cdot c = c \cdot b = 1.$$

This number, the *multiplicative inverse* of b, is $c = b^{-1} = 1/b$, the *reciprocal* of b.

We can extend these ideas from numbers to functions. There is exactly one function $k : \mathbf{R} \to \mathbf{R}$ such that

$$k(x) \cdot f(x) = f(x) \cdot k(x) = f(x), \quad \text{for all } f : \mathbf{R} \to \mathbf{R}$$

This function, the *multiplicative identity* for functions, has the formula $k(x) = 1$, for all $x \in \mathbf{R}$.

The *multiplicative inverse* of the function f is the function g such that

$$f(x) \cdot g(x) = g(x) \cdot f(x) = k(x) = 1.$$

In fact, $g(x) = [f(x)]^{-1} = 1/f(x)$, which exists for all x in the domain of f for which $f(x) \neq 0$.

Composition of Functions

This operation has no counterpart for real numbers. Recall that $(f \circ g)(x) = f(g(x))$. The *identity function* (under composition) is the function $\iota : \mathbf{R} \to \mathbf{R}$ such that

$$(f \circ \iota)(x) = (\iota \circ f)(x) = f(x), \quad \text{for all } f : \mathbf{R} \to \mathbf{R}.$$

The formula for ι is $\iota(x) = x$, for all $x \in \mathbf{R}$.

The function g is the *inverse* of f (under composition) if

$$(f \circ g)(x) = (g \circ f)(x) = \iota(x) = x,$$

That is, if g is the inverse of f under composition, then $f(g(x)) = g(f(x)) = x$ for all x in the domain of f. The inverse of f is generally denoted by $f^{-1}(x)$.

NOTE: In general, the multiplicative inverse of f is not 'the inverse "of f:

$$[f(x)]^{-1} \neq f^{-1}(x)$$

Example

The natural logarithm $g(x) = \ln(x)$ is the inverse of the exponential function $f(x) = e^x$:

$$e^{\ln(x)} = \ln(e^x) = x \quad \text{but} \quad \ln(x) \neq \frac{1}{e^x} = (e^x)^{-1}.$$

Graphs of Inverse Functions

1. Suppose $f(a) = b$. (This means that the point (a, b) is on the graph of f.)
   Show that $f^{-1}(b) = a$. (This means that the point (b, a) is on the graph of f^{-1}.

2. Use this result to explain why the graph of f^{-1} is the *reflection about the line* $y = x$ of the graph of f.

Example: $f(x) = e^x$, $-\infty < x < +\infty$,

$f^{-1}(x) = \ln(x) \; - \infty < x < +\infty$.

Example: $g(x) = \sin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$,

$g^{-1}(x) = \arcsin(x) := \sin^{-1}(x)$, $-1 \leq x \leq 1$.

5. Does *every* function have an inverse? Consider $h(x) = \sin(x)$, $-\pi \leq x \leq \pi$.

Derivatives of Inverse Functions

Analytic Approach

If $g = f^{-1}$, then $x = f(g(x))$. Then by the Chain Rule,

$$1 = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \quad \text{so} \quad g'(x) = \frac{1}{f'(g(x))},$$

provided $f'(g(x)) \neq 0$. In the usual notation for inverse functions,

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}, \quad \text{provided } f'(f^{-1}(x)) \neq 0.$$

Example: For $-1 \le x \le 1,\ x = \sin(\arcsin(x))$ and $\frac{d}{dx}\sin(x) = \cos(x) = \sqrt{1 - \sin^2(x)}$, so

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

Graphical Approach

The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

The line tangent to the graph of f^{-1} at (b, a) is the reflection across the line $y = x$ of the line tangent to the graph of f at (a, b).

If (d, c) is on the line tangent to the graph of f^{-1} at (b, a), then (c, d) is on the line tangent to the graph of f at (a, b).

Thus if (d, c) is another point on the line tangent to the graph of f^{-1} at (b, a), the slope of this line can be computed as

$$(f^{-1})'(b) = \frac{a - c}{b - d} = 1\Big/\left(\frac{b - d}{a - c}\right) = 1/f'(a),$$

the reciprocal of slope of the line tangent to the graph of f at (a, b).

More Examples of Derivatives of Inverse Functions

6. The inverse of the function $f(x) = \tan(x)$ $-\pi/2 \leq x \leq \pi/2$ is the function $g(x) = \arctan(x)$, $-\infty \leq x \leq \infty$. Starting with

$$\tan(\arctan(x)) = x, \quad -\infty < x < \infty,$$

use the Chain Rule to find $\frac{d}{dx}\arctan(x)$. To simplify your answer, it will be useful to express the derivative of $\tan(x) = \sin(x)/\cos(x)$ in terms of $\tan(x)$.

Inverse Functions and Antiderivatives

7. Every differentiation rule implies an antidifferentiation rule! Use the work we have just completed to find the following antiderivatives:

$$\int \frac{dx}{x} =$$

$$\int \frac{dx}{\sqrt{1-x^2}} =$$

$$\int \frac{dx}{1+x^2} =$$

8. Evaluate the following definite integrals:

$$\int_{-0.5}^{0.5} \frac{dx}{\sqrt{1-x^2}} =$$

$$\int_{0}^{\pi/2} \frac{\sin y dy}{1+\cos^2 y} =$$