Introduction to Taylor Series

You are familiar with Taylor polynomials (at least of first and second degree) from Math 110. Recall that we used these as polynomials which did a good job of approximating a given function near a given point.

More recently, you have also become familiar with infinite series.

Now we are going to put these two topics together. We are going to find infinite series, developed as powers of x, which actually equal a given function on some interval about a given point. These series – Taylor series – come from extending the idea of Taylor polynomials from first and second degrees to all degreees.

Definition (Taylor Series)

Let f be a function which has derivatives of all orders at the point a. Then the Taylor series for f about a is defined to be the infinite series:

$$\sum_{k=0}^{\infty} c_k (x-a)^k, \qquad \text{where } c_k = f^{(k)}(a)/k!.$$

Notation:

$f^{(k)}$ denotes the kth derivative of f. Thus, $f^{(k)}(a)$ denotes the kth derivative of f evaluated at the point a.

By convention, $f^{(0)}$ denotes the function f itself. Thus, $f^{(0)}(a) = f(a)$.

$k!$, read "k factorial," denotes the product $1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$ of the positive integers 1 through k.

Example 1: The Taylor series for $f(x) = \sin(x)$ about $a = 0$

$$\begin{array}{rcl} f(x) = \sin(x) & & f(0) = \sin(0) = 0 & & c_0 = f(0)/0! = 0 \\ f'(x) = \cos(x) & & f'(0) = \cos(0) = 1 & & c_1 = f'(0)/1! = 1 \\ f''(x) = -\sin(x) & & f''(0) = -\sin(0) = 0 & & c_2 = f''(0)/2! = 0 \\ f^{(3)}(x) = -\cos(x) & \Longrightarrow & f^{(3)}(0) = -\cos(0) = -1 & \Longrightarrow & c_3 = f^{(3)}(0)/3! = -1/3! \\ f^{(4)}(x) = \sin(x) & & f^{(4)}(0) = \sin(0) = 0 & & c_4 = f^{(4)}(0)/4! = 0 \\ \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots \end{array}$$

Then the Taylor series for $\sin(x)$ about $a = 0$ is given by

$$c_0 + c_1(x-0) + c_2(x-0)^2 + c_3(x-0)^3 + c_4(x-0)^4 + ... = x - \frac{1}{3!}x^3 + \ldots$$

In fact, as this pattern continues, we can write this series as

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}.$$

Problem 1

Follow the previous example to find Taylor series for $f(x) = \frac{1}{1-x}$ about $a = 0$. Find the terms for this series explicitly through the fourth degree, then guess at the general series. Do you recognize this series? Use the Absolute Ratio Test to determine the values of x for which this series converges. (This is called the interval of convergence.)

Problem 2

Similarly, find the Taylor series for $f(x) = e^x$ about $a = 0$. Find the terms for this series explicitly through the fourth degree, then guess at the general series. Use the Absolute Ratio Test to determine the values of x for which this series converges. (This is called the interval of convergence.)

Taylor Polynomials and Derive

You can use *Derive* to easily find Taylor polynomials of different degrees. Select `Calculus` then `Taylor series` from the menu at the top of the screen, then fill in the appropriate information in the dialogue box that appears. Choose `Simplify` rather than `OK` to get the polynomial. Alternatively, you can choose `Simplify` from the menu bar afterwards.

Use this procedure to find Taylor polynomials of degrees 1, 2, 5, 10, 20 and 40 for $f(x) = 1/(1-x)$ about $a = 0$. Plot these together on the same plot. What happens as you increase the degree of the polynomials? How is this related to the interval of convergence?

Use this same procedure to find Taylor polynomials of degrees 1, 2, 5, 10, 20 and 40 for $f(x) = exp(x)$ about $a = 0$. Plot these together on the same plot. (Clear your plot of the previous graphs first.) What happens as you increase the degree of the polynomials? How is this related to the interval of convergence?