This lab is to concerned with the technique of transforming difficult antiderivatives and integrals into more easily evaluated ones using the technique of cleverly chosen "substitution functions". There are two goals for this lab. The first is to gain more practice with this technique and to use *Derive* to check your results. The second is to gain some geometric understanding of how this technique transforms the integrand and the interval of integration.

1. Antiderivatives

Consider the two functions $f(x)$ and $h(u)$ where

$$f(x) = \frac{3x^2}{\sqrt{x^3+1}}, \qquad h(u) = \frac{1}{\sqrt{u}}.$$

Use *Derive* to carefully plot $f(x)$ on the interval $1 \leq x \leq 2$. Sketch this plot below.

Use *Derive* to carefully plot $h(u)$ on the interval $2 \leq x \leq 9$. Sketch this plot below.

(a) In what way are these graphs similar? How do they differ? What would you have to do *geometrically* to transform the graph of f into the graph of h?

Use *Derive* to evaluate: $\displaystyle\int \frac{3x^2}{\sqrt{x^3+1}}\,dx =$

Use *Derive* to evaluate: $\displaystyle\int \frac{1}{\sqrt{u}}\,du =$

(b) Compare these two antiderivatives. How are they related?

(c) By hand, use the substitution $u = g(x) = x^3 + 1$ to simplify, then determine the following antiderivative:

$$\int \frac{3x^2}{\sqrt{x^3+1}}\,dx =$$

2. Definite Integrals

Now transform the limits of integration as well, then evaluate the final definite integral.

$$\int_1^2 \frac{3x^2}{\sqrt{x^3+1}\,dx} = \int \frac{1}{\sqrt{u}}\,du =$$

Check your results with *Derive*.

Also, on your plots on the first page of this lab shade the areas under the graphs of $f(x)$ and $h(u)$ equal to the two definite integrals above.

3. Some More Antiderivatives

To do this problem you will need to "recall" a number of trigonometric identities. Here they are:

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 1 - 2\sin^2(x) \\
&= 2\cos^2(x) - 1
\end{aligned}$$

(a) Confirm that the functions $\cos(x)\sin(x)$ and $\frac{1}{2}\sin(2x)$ are indeed identical by using *Derive* to plot them on the same axes.

(b) Evaluate $\int \sin(x)\cos(x)\,dx$. [Use the substitution function $u = \cos(x)$]

(c) Evaluate $\int \sin(x)\cos(x)\,dx$. [Use the substitution function $u = \sin(x)$]

(d) Evaluate $\int \frac{1}{2}\sin(2x)\,dx$.

(e) Are your answers to **(b)**,**(c)** and **(d)** the same? If not, why not? Remember, these are *antiderivatives.*

(f) Use the trigonometric identities to obtain relationships between the arbitrary constants in **(a)**, **(b)** and **(c)**.

[HINT: Represent each arbitrary constant by a different letter, then use the trig identities above to obtain relationships between the constants]

4. More Definite Integrals

In this problem we will consider how to use integration by substitution on definite integrals. You will see that there is more than one u substitution that can be used to get the correct answer.

(a) Use u-substitution to rewrite, then evaluate $\int_0^{\pi/2} \frac{1}{2}\sin(2x)\,dx$ by hand. [Use *Derive* to check your answer.]

(b) Use u-substitution to rewrite, then evaluate $\int_0^{\pi/2} \sin(x)\cos(x)\,dx$. Check with *Derive*.

(c) Use u-substitution to rewrite, then evaluate $\int_0^{\pi/2} \sin(x)\cos(x)\,dx$. Check with *Derive*.

(d) Are your final results in **(a)**, **(b)** and **(c)** identical? Should they be?

No Report Is Necessary for this Lab

While no report is necessary for this lab, you should be able to answer the following questions based on it. We suggest you discuss these with your lab group.

Consider the integrals investigated in numbers **1.** and 2. above. Geometrically, how did substitution transform the integrand f? How does this geometric transformation relate to each of the substitution steps? For each of the two *definite* integrals, determine the smallest number of subdivisions required for the difference between left- and right-hand Riemann sums approximations to be less than 0.1. Does it take the same number for both integrals? Try to explain your result.

Use your results from **3.** and **4.** to comment on the relationships between the area under a $\sin^2(x)$ curve (or a $\cos^2(x)$ curve) between $x = 0$ and $x = \pi/2$ and the area under a parabola $y = x^2$ between $x = 0$ and $x = 1$. Give a sketch.

Part 3: With the knowledge gained from this lab and other work in this course on substitution, evaluate these (apparently difficult) definite integrals by hand. Show all your work.

(a) $\int 3x^2(x^3+1)^{49.3}\,dx$

(b) $\int_0^1 3x^2[4(x^3+2)^{29} - \sin(\pi(x^3+2)) + e^{x^3+2}]\,dx$

(c) $\int_0^1 \sqrt{1+\sqrt{1+\sqrt{x}}}\,dx$