Definite Integrals

Many scientific problems can be solved by the following strategy:

Subdivide-Approximate-Accumulate-Refine

Examples we have studied include determining *area*, *volume*, *distance traveled*, *human work done* and *electrical energy consumed*. Let Q be such a quantity. Often the subdivisions ΔQ_k can be approximated by terms of the form

$$\Delta Q_k \approx f(t_k)\Delta t_k, \qquad a \le t_0 < \cdots < t_n \le b, \quad \Delta t_k = t_k - t_{k-1},$$

with f at least *piecewise continuous* on the interval $[a, b]$. For volume, f is the cross-sectional area as a function of distance from the end of the object; for distance traveled, f is the velocity as a function of time; for human work done, f is the staffing level as a function of time; and for electrical enery consumed, f is the electrical power as a function of time.

Accumulating these subdivision approximations yields a **Riemann sum**. The sum is a *left-hand sum* (LHS) if f is evaluated at the left-endpoint of each subinterval. It is a *right-hand sum* (RHS) if f is evaluated at the right-endpoint of each subinterval. (If the function is evaluated at other points in the subinterval no special name is given to the Riemann sum.)

$$\text{(LHS):} \quad \sum_{k=0}^{n-1} f(t_k)\Delta t_k, \qquad \text{(RHS):} \quad \sum_{k=1}^{n} f(t_k)\Delta t_k, \qquad a \le t_0 < \cdots < t_n \le b.$$

If f is piecewise continuous and/or monotone, refining the subdivisions so that $\Delta t_k \to 0$ as $n \to \infty$ causes these sums to converge to the same value, a **definite integral**:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(t_k)\Delta t_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(t_k)\Delta t_k = \int_a^b f(t)\,dt.$$

A useful interpretation of this definite integral is as the *signed area* between the graph of $f(t)$ and the t axis over the interval $[a, b]$. Area below the axis is regarded as negative in this interpretation.

f is **monotone increasing** on $[a, b]$ if it does not decrease on this interval. In this case,

$$\text{LHS} \le \int_a^b f(t)\,dt \le \text{RHS} \quad \text{and} \quad |\text{RHS} - \text{LHS}| = |f(b) - f(a)| \cdot \Delta t, \quad \Delta t = (b-a)/n.$$

f is **monotone decreasing** on $[a, b]$ if it does not increase on this interval. In this case,

$$\text{RHS} \le \int_a^b f(t)\,dt \le \text{LHS} \quad \text{and} \quad |\text{RHS} - \text{LHS}| = |f(b) - f(a)| \cdot \Delta t, \quad \Delta t = (b-a)/n.$$

Thus the **accuracy** of Riemann sum approximations to integrals of montone functions and the number n of subdivisions needed to achieve a certain accuracy can be calculated.

Using both the definition and geometric interpretations, it is easy to establish the following **properties of definite integrals**:

$$\int_a^b f(t)\,dt + \int_a^b g(t)\,dt = \int_a^b f(t)+g(t)\,dt, \qquad \int_a^b c\cdot f(t)dt = c\cdot\int_a^b f(t)dt, \quad \text{constant } c,$$

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt, \qquad \int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$$

Accumulation Functions and the Fundamental Theorem of Calculus (FTC)

A definite integral evaluates to a single number. However, if we allow the upper limit of the interval of integration to vary we get a function called an **accumulation function**:

$$F(T) = \int_a^T f(x)dx.$$

Thinking about Riemann sum approximations to the integral with fixed T, it is easy to see that $F(T+\Delta T) - F(T) \approx f(T)\Delta T$. From this follows one version of the **FTC**:

$$F'(T) = \frac{d}{dT}\left(\int_a^T f(x)dx\right) = f(T), \qquad \text{if } f \text{ is continuous at } T.$$

Observing that $F(a) = 0$, we have another version of the **FTC**:

$$y(T) = \int_a^T f(t)\,dt \quad \text{is the solution of} \quad \begin{cases} y'(T) = f(T) \\ y(a) = 0 \end{cases}$$

provided f is continuous at T.

This theorem is also illustrated by looking carefully at Euler's method for this initial value problem and left-hand Riemann sums for the accumulation function. They are identical!

Antiderivatives and the Fundamental Theorem of Calculus

If $F'(x) = f(x)$, then F is said to be an **antiderivative** of f. The antiderivative of f is not unique, but two such antiderivatives differ only by an added constant. The last version of the **FTC** states that if F is an antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

An antiderivative of a continous function f can always be written as an accumulation function. While this can always be approximately evaluated numerically, it is *sometimes* possible to find a formula for an antiderivative in closed form. In seeking closed forms remember the following: **Every differentiation rule implies an antidifferentiation rule.**

Thus the basic derivative formulas you know well provide basic antiderivative formulas. More complicated formulas are found in tables or through symbolic computing packages like *Derive*.