Partial Derivatives for a Function of Two Variables

Partial Derivatives

If you are at a given point (a, b) in the domain of a function of two variable, there are in general *infinitely many different directions* one could move and still be in the domain of the function. You can seek the rate of change of the function as you move in any of these directions.

To keep things simple at first, we will consider only moving in directions parallel to the x or y axes.

1. Suppose you are at the point (a, b) in the domain of a function $f(x, y)$ of two variables. Suppose you move Δx units parallel to the x axis. What is the x-coordinate of the point you are at then? Has the y-coordinate changed? What is it?

The average rate of change of f between these two points is

$$\frac{\Delta z}{\Delta x} = \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}.$$

If we let Δx get smaller and smaller, the point $(a + \Delta x, b)$ approaches the point (a, b). If the difference quotient above approaches a definite limit as $\Delta x \to 0$, then we define that as:

DEFINITION:

- **The partial derivative of f with respect to x, at the point** (a, b), is defined as the following limit, if it exists:
$$\frac{\partial z}{\partial x} = f_x(a, b) := \lim_{\Delta x \to 0} \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}.$$
- Similarly, **the partial derivative of f with respect to y, at the point** (a, b), is defined as the following limit, if it exists:
$$\frac{\partial z}{\partial y} = f_y(a, b) := \lim_{\Delta y \to 0} \frac{f(a, b + \Delta y) - f(a, b)}{\Delta y}.$$

  Note that there are two sorts of notations commonly used for partial derivatives.

2. Compare the definitions of $f_x(a, b)$ and $f_y(a, b)$. Explain why $f_y(a, b)$ is defined by considering points near (a, b) along a line parallel to the y-axis.

Calculating Partial Derivatives

Fortunately, we rarely actually have to evaluate a limit to compute a partial derivative. Your knowledge of derivatives for functions of one variable, along with the idea of vertical slices, is enough! Recall that we defined the vertical slice of f along x, holding $y = b$, as $g(x) = f(x, b)$. Therefore,

$$f_x(a,b) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x, b) - f(a,b)}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(a + \Delta x) - g(a)}{\Delta x} = g'(a).$$

TO COMPUTE $f_x(a,b)$, JUST DIFFERENTIATE $f(x,y)$ WITH RESPECT TO x, regarding it as a function of x only (i.e. TREATING y AS IF IT WERE A CONSTANT), then evaluate the result at $x = a$, $y = b$.

Similarly, we defined the vertical slice of f along y, holding $x = a$, as $\phi(y) = f(a, y)$. Therefore,

$$f_y(a,b) = \lim_{\Delta y \to 0} \frac{f(a, b + \Delta y) - f(a,b)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\phi(b + \Delta y) - \phi(b)}{\Delta y} = \phi'(b).$$

TO COMPUTE $f_y(a,b)$, JUST DIFFERENTIATE $f(x,y)$ WITH RESPECT TO y, regarding it as a function of y only (i.e. TREATING x AS IF IT WERE A CONSTANT), then evaluate the result at $x = a$, $y = b$.

The problems below concern the function $f : \mathbb{R}^2 \to \mathbb{R}, \quad z = f(x,y) = 2x^2 + 3xy - 4.$

3. Find the following partial derivatives:

$f_x(2,3)$

$f_y(2,3)$

$f_x(a,b)$

$f_y(a,b)$

$f_x(x,y)$

$f_y(x,y)$

More Practice With Partial Differentiation

Examples

4. For each multivariable function $f(x, y)$ below find the partial derivative with respect to x, $f_x(x, y)$ and the partial derivative with respect to y. Set both partial derivatives equal to zero at the same time. Are there any locations in the xy-plane where $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously?

a. $g(x, y) = x^2 + y^2$

b. $j(x, y) = x^2 - y^2$

c. $f(x, y) = (x + y)^2$

d. $k(x, y) = \sin(x) - \sin(y)$

5. Proof that partial differentiation can be easier than "ordinary" differentiation:
Evaluate $\dfrac{\partial}{\partial y} e^{\sin(x^2 + 2^x)}$ (Be careful!!)