Indeterminate Forms and L'Hôpital's Rule

Limits and the Indeterminate Form $0/0$

We know that $f(x) = \sin(x)$ and $g(x) = x$ are continuous everywhere, so why isn't it true that:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \neq \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}?$$

When it appears as the apparent result of taking a limit, $0/0$ is considered an *indeterminate form*. This language is used to indicate that this value doesn't determine whether or not a limit value exists; if a different technique is used, that limiting value may be found. Thus, in an indeterminate case, further work is required.

Taylor's Theorem (sound familiar?) is the key in many cases to determining whether or not the limit exists. To apply this idea, expand both the numerator and denominator in a first-degree Taylor polynomial about the limit point, using Taylor's Theorem to include the error term. Then simplify and take the limit.

What does Taylor's Theorem say?

Example:

Let $f(x) = \sin(x)$ and $g(x) = x$. Using Taylor's Theorem, we can write

$f(0 + h) =$

$g(0 + h) =$

Now we can consider the limit. (Hint: use the fact that the limit of the relative errors goes to zero.)

$\lim_{h\to 0} \frac{f(h)}{g(h)} =$

If we look carefully at how this example was handled, we can obtain a more general result which will save time and calculation. Suppose that f and g are both differentiable at a and that $f(a) = 0 = g(a)$. Set $x = a + h$ so that $h = x - a$. Then by Taylor's Theorem,

$\frac{f(x)}{g(x)} =$

Therefore, if $f'(a)/g'(a)$ exists, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

Note that in case $f'(x)$ and $g'(x)$ are continuous at a, then $\lim\limits_{x\to a} f'(x)/g'(x) = f'(a)/g'(a)$, provided this makes sense. *When does this make sense?*

This result, extended to slightly more general circumstances, is summarized in the following theorem:

L'Hôpital's Rule:

Suppose $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$, and that

1. f and g are differentiable at a,
2. $g'(x) \neq 0$ for $x \neq a$ on some open interval containing a,
3. and $\lim_{x\to a} f'(x)/g'(x)$ is well-defined (as a finite value, $+\infty$ or $-\infty$).

Then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

L'Hôpital's Rule is a very useful tool in the evaluation of limits. We can also use it in the case of one-sided limits with the obvious modifications. The extra conditions in this theorem admit the possibility that you still have an indeterminate form after applying L'Hospital's Rule once. You may need to apply it several times before you can resolve the limit.

Examples: Use L'Hôpital's Rule to evaluate the following limits:

$$\lim_{h\to 0}\frac{1-\cos(h)}{h}$$

$$\lim_{t\to 0}\frac{e^t-1-t}{t^2}$$

Infinity as a Limiting Value: Suppose that for every finite number $M > 0$, we know that $f(x) > M$ for all x sufficiently close to a.

What does this mean? Since we can make M as large as we please, this means that $f(x)$ grows without bound as $x \to a$. In mathematical words,

$$\lim_{x \to a} f(x) = +\infty.$$

Write your own, similar definition definition for $\lim_{x \to a} f(x) = -\infty$:

It is often useful to keep in mind the following:

If the following conditions hold:

1. $\lim_{x \to a} f(x) = c > 0$,
2. $g(x) > 0$ for $x \neq a$ on some open interval containing a,
3. and $\lim_{x \to a} g(x) = 0$.

Then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = +\infty$.

There is a similar statement regarding limits tending towards $-\infty$. As with finite limits, we say that the limit is $+\infty$ *if and only if* the limits from above and below are **both** $+\infty$. Similarly, the limit is $-\infty$ *if and only if* the limits from above and below are **both** $-\infty$.

Example: Determine $\lim_{x \to 0^+} 1/x$ and $\lim_{x \to 0^-} 1/x$. What do these results imply about $\lim_{x \to 0} 1/x$?

Limits and the Indeterminate Form ∞/∞

If $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = +\infty$, then the limit $\lim_{x\to a} f(x)/g(x)$ is indeterminate. We will not prove this as we did earlier for the case 0/0, but L'Hôpital's Rule can be extended to this case. (Hooray!)

L'Hôpital's Rule (II)

Suppose $\lim_{x\to a} f(x) = +\infty = \lim_{x\to a} g(x)$, and that

- f and g are differentiable at a,
- and $\lim_{x\to a} f'(x)/g'(x)$ is well-defined (as a finite value, $+\infty$ or $-\infty$).

Then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Once again, we can apply this in the case of one-sided limits with the obvious modifications.

Example: Evaluate $\displaystyle\lim_{x\to 0^+} \frac{\ln(x)}{e^{1/x}}$.

Limits at Infinity: What does it mean if $\lim_{x\to\infty} f(x) = L$?

This means that as we let x get larger and larger, $f(x)$ gets as close as we please to L. In the case, $\lim_{x\to-\infty} f(x) = L$, we are considering what happens as we let x get smaller and smaller. It is helpful to use the following fact when evaluating limits at infinity:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to 0^+} f(\frac{1}{x}).$$

Example: Evaluate $\lim_{x\to+\infty} \sin(1/x)$.