Derivatives of Inverse Functions

Warm-up

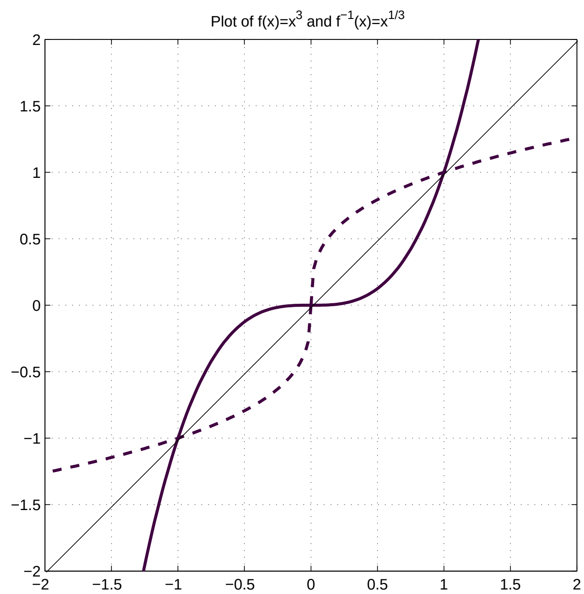
Given two functions $f(x)$ and $g(x)$ which have the relationship that $f(g(x)) = x$ and $g(f(x)) = x$ use the Chain Rule to show that $f'(g(x)) = \dfrac{1}{g'(x)}$ and $g'(f(x)) = \dfrac{1}{f'(x)}$. *What is a word that we could use to describe $f(x)$ and $g(x)$?*

Derivative of the Inverse Function

If $g = f^{-1}$, then $x = f(g(x))$. In the usual notation for inverse functions,

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}, \quad \text{provided } f'(f^{-1}(x)) \neq 0.$$

Graphical Approach



The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.
The line tangent to the graph of f^{-1} at (b, a) is the reflection across the line $y = x$ of the line tangent to the graph of f at (a, b).

If $(d, c) \neq (b, a)$ is on the line tangent to the graph of f^{-1} at (b, a), then $(c, d) \neq (a, b)$ is on the line tangent to the graph of f at (a, b).

The derivative of f^{-1} at b can then be computed as the slope of the line tangent to the graph of f^{-1} at (b, a):

$$(f^{-1})'(b) = \frac{a - c}{b - d} = 1 \Big/ \left(\frac{b - d}{a - c} \right) = 1/f'(a),$$

the reciprocal of the slope of the line tangent to the graph of f at (a, b).

Examples of the Derivative of an Inverse Function

1. The inverse of the function $f(x) = \tan(x)$, $-\pi/2 \leq x \leq \pi/2$
is the function $g(x) = \arctan(x)$, $-\infty < x < +\infty$.
Starting with

$$\tan(\arctan(x)) = x, \quad -\infty < x < +\infty,$$

use the Chain Rule to find $\dfrac{d}{dx}\arctan(x)$.
To simplify your answer, it will be useful to express the derivative of $\tan(x) = \sin(x)/\cos(x)$ in terms of $\tan(x)$, using the identity $1 + \tan^2 x = \sec^2 x$.
(Extra: prove this identity holds by starting with the identity $\cos^2 x + \sin^2 x = 1$.)

2. The inverse of the function $f(x) = e^x$, $-\infty < x < +\infty$
is the function $g(x) = \ln(x)$, $0 < x < \infty$.
Starting with

$$e^{\ln(x)} = x, \quad 0 < x < \infty,$$

use the Chain Rule to find $\dfrac{d}{dx}\ln(x)$.

The Power Rule for Real Exponents

We can also use the Chain Rule similarly to extend the power rule to its most general form:

THEOREM: If r is any real number and $x > 0$, then $\frac{d}{dx}x^r = r\,x^{r-1}$.

Proof:

If $x > 0$, then $x^r = e^{r\ln x}$; this follows from two properties of exponential and logarithmic functions:

$$y > 0 \implies e^{\ln y} = y, \quad \text{and } x > 0 \implies \ln x^r = r\ln x.$$

Then by the Chain Rule,

$$\begin{aligned} \frac{d}{dx}x^r &= \tfrac{d}{dx}e^{r\ln x} \\ &= e^{r\ln x}\tfrac{d}{dx}r\ln x \\ &= x^r \cdot \tfrac{r}{x} \\ &= r\,x^{r-1}. \end{aligned}$$

Examples

If $f(x) = \frac{1}{x^\pi}$, write down $f'(x)$.

Given $y = x^{\sqrt{2}}$, find $\frac{dx}{dy}$ and $\frac{dy}{dx}$