The Algebra and Geometry of Inverse Functions

Inverses and Identities

Many operations on sets of numbers or functions have an *identity element* as well as *inverses* in the set. Important examples include addition, multiplication, and composition of functions.

1. Addition:

(i.) There is exactly one real number a with the property that

$$a + x = x + a = x$$

for all $x \in \mathbf{R}$. This number is called the **additive identity**. The additive identity $a =$ ______.

(ii.) If b is a real number, there is exactly one real number c such that

$$b + c = c + b = 0.$$

This number, c is called the **additive inverse of b**. For b, the additive inverse is $b =$______.

(iii.) We can extend these ideas from numbers to functions. There is exactly one function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) + f(x) = f(x) + h(x) = f(x), \quad \text{for all } f : \mathbb{R} \to \mathbb{R}.$$

What does the notation, $f : \mathbb{R} \to \mathbb{R}$ mean?

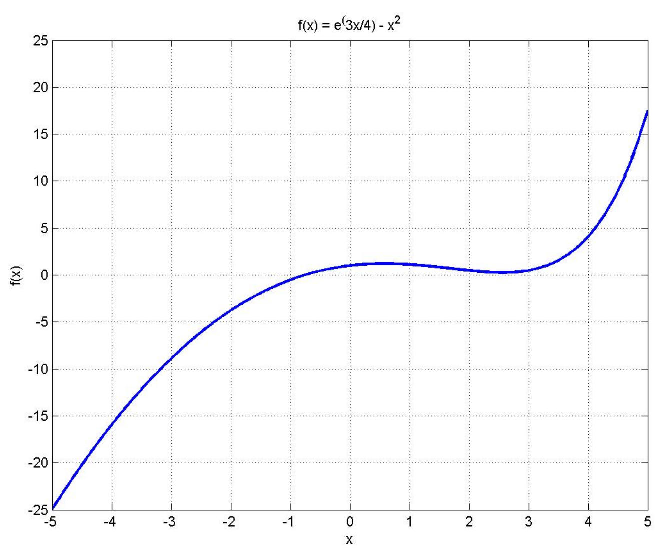
This function, $h(x)$ is the ____________________ **for** $f(x)$. The formula for $h(x)$ is $h(x) =$ ______, for all $x \in \mathbb{R}$.

(iv.) For a function $f : \mathbb{R} \to \mathbb{R}$, there is a function $g : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) + g(x) = g(x) + f(x) = h(x) = 0.$$

The function $g(x)$ is called the ____________________ for the function $f(x)$. In fact, $g(x) =$____________________. The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ across the ______-axis.

On the graph below sketch the graph of $-f(x)$.



Multiplication

(i) There is exactly one real number a with the property that

$$a \cdot x = x \cdot a = x, \quad \text{for all } x \in \mathbb{R}.$$

This number is the ________________________; $a =$ ______.

(ii) If $b \neq 0$ is a real number , there is exactly one real number c such that

$$b \cdot c = c \cdot b = 1.$$

This number is called the ________________________ of b. It is written $c =$ ____________ and is also known as the ________________________ of b.

(iii) Again, we can extend these ideas from numbers to functions. There is exactly one function $k : \mathbb{R} \to \mathbb{R}$ such that

$$k(x) \cdot f(x) = f(x) \cdot k(x) = f(x), \quad \text{for all } f : \mathbb{R} \to \mathbb{R}$$

This function is the ________________________ and has the formula $k(x) =$ ____________ for all $x \in \mathbb{R}$.

(iv) Given a function $f : \mathbb{R} \to \mathbb{R}$, there is a function $g : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) \cdot g(x) = g(x) \cdot f(x) = k(x) = 1.$$

In fact, $g(x) =$________________________. *When does this function exist?*

Composition of Functions

We don't speak of composing real numbers; this is an operation particular to functions.

*(i)*With respect to composition of functions, there is a special function $I : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ I)(x) = (I \circ f)(x) = f(x), \quad \text{for all } f : \mathbb{R} \to \mathbb{R}.$$

This function is called the ____________________________________. $I(x) =$________________________ for all $x \in \mathbb{R}$.

(ii) There are also functions, $g(x)$ so that, under composition with a function $f(x)$,

$$(f \circ g)(x) = (g \circ f)(x) = I(x) = x.$$

The function $g(x)$ is called the ________________________ of $f(x)$. That is, if g is the inverse of f under composition, then $f(g(x)) = g(f(x)) = x$ for all x in the domain of f. The inverse of f is generally denoted by $f^{-1}(x)$. *When does this function exist?*

Question: In general, is the multiplicative inverse of f equal to its inverse under composition, "the inverse" function?

Examples:

(1) Let $f(x) = 2x$. What is its multiplicative inverse? What is its inverse function?

(2) Let $f(x) = \ln(x)$. What is its multiplicative inverse? What is its inverse function?

Graphs of Inverse Functions

Suppose $f(a) = b$. (This means that the point (a, b) is on the graph of f.)
Show that $f^{-1}(b) = a$. (This means that the point (b, a) is on the graph of f^{-1}.)

Use this result to explain why the graph of f^{-1} is the *reflection about the line* $y = x$ of the graph of f.

Examples: Verify that the following pairs of functions are inverses and sketch the function and its inverse on the same set of axes.

(a) $f(x) = e^x$, $f^{-1}(x) = \ln(x)$.

(b) $g(x) = \tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad g^{-1}(x) = \arctan(x) := \tan^{-1}(x), -1 < x < 1.$

Does *every* function have an inverse? Consider $h(x) = \sin(x), -\pi \leq x \leq \pi$.