Composition of Functions and the Chain Rule

Warm-Up

1. Find the *natural domain* and *range* for functions with the following formulas:

(a) $y = f(x) = 1 - |x|$

(b) $z = g(y) = \ln(y)$

(c) $w = h(x) = g(f(x))$

(d) $u = p(x) = -|x|$

(e) $v = q(x) = g(p(x))$

Composition of Functions

Suppose $y = f(x)$ and $z = g(y)$ are two real-valued functions. If the intersection of the set of possible *output* values of f and the set of possible *input* values of g, i.e.

$$(\text{Range of } f) \cap (\text{Domain of } g)$$

is not empty, then the **composition of** g **with** f, denoted by the symbols $g \circ f$, is defined as

$$z = (g \circ f)(x) = g(f(x)).$$

Similarly, $f \circ g = f(g(x))$.

The **domain of** $g \circ f$ is $\{\, x \in \text{Domain of } f \mid f(x) \in \text{Domain of } g\}$. In other words, the domain of the composition of g with f is all x values in the domain of f which produce $f(x)$ values such that (denoted by "—") these $f(x)$ values are in the domain of g.

2. Which of the functions above in the *Warm-Up* question are explicitly defined as the composition of two functions?

3. What are the domain of these composition functions?

4. Can any of the other functions be defined as the composition of two functions? How?

The Chain Rule

Suppose $y = f(x)$ is differentiable at $x = a$ and $z = g(y)$ is differentiable at $y = f(a)$. Then the composition $z = g \circ f(x)$ is differentiable at $x = a$, and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

The Chain Rule is a powerful theoretical tool as well as being useful in calculating derivatives of particular functions.

Interpreting The Chain Rule
Suppose z is a function of y, and y is a function of x.
Suppose we're given the following:
(a) The rate of change of z with respect to y is 4.
(b) The rate of change of y with respect to x is 3.
Part (a) says: For every unit increase in y, z increases by ____ units.
Part (b) says: For every unit increase in x, y increases by ____ units.

THEREFORE, if we increase x by 1 unit, y will increase by ____ units, which in turn causes z to increase by ____ units.

We could also use the **Microscope Approximation** to interpret the Chain Rule

Suppose $f(g(x))$is a composite function. Let us write

$$y = f(u) \text{ and } u = g(x), \text{ so } y = f(g(x)).$$

The function $g(x)$ is called the *inner function* and f is called the *outer function*.
Write the microscope approximation for $u = g(x)$ at a point x. (How does a change in x effect a change in u?)

Write the microscope approximation for $y = f(u)$ at a point u. (How does a change in u effect a change in y?)

Combine these two microscope approximations to relate a change in y to a change in x.

We can also think of the Chain Rule using **differentials**
On the tangent line to the graph of $y = f(x)$ at $(a, f(a))$: $\quad dy = f'(a)dx$

On the tangent line to the graph of $z = g(y)$ at $(f(a), g(f(a)))$: $\quad dz = g'(f(a))dy$

If we set the output dy on the tangent line for f equal to the input dy on the tangent line for g, then

$$dz = g'(f(a))f'(a)dx.$$

5. If $r(y) = y^{-1}$, what is $r'(y)$?

6. How does the above observation complete the derivation of the Quotient Rule in *Class 15*?

Practice with the Chain Rule

7. Find derivatives, at the specified points, of the composite functions with the formulas given below. In each case, **explicitly state the functions making up the composition** and apply the Chain Rule.

a) $z = (g \circ f)(x) = \sqrt{1 - x^2}$

b) $t = (q \circ p)(r) = 3^{6\,r}$

c) $w = (h \circ \phi)(u) = \sin(10u + 3)$

d) $s = (k \circ n)(y) = \dfrac{2}{y^2 + y + 1}$