Taylor's Theorem

Recall all the linearization we have done for functions: we looked at locally linear functions with Euler's Method, tangent line approximations and the Microscope Approximation. We will use the Microscope Approximation in order to see Taylor's Theorem today and that will lead us to the product and quotient rules in the next class!

On the Error in the Microscope Approximation

You know that if f is differentiable at the input value a, then the *Microscope Approximation* gives you a way to approximate $f(x)$ for x near a:

$$\begin{aligned} \Delta y &\approx f'(a)\Delta x \\ f(a+\Delta x) - f(a) &\approx f'(a)\Delta x \\ f(a+\Delta x) &\approx f(a) + f'(a)\Delta x \end{aligned} \qquad \text{or} \qquad \begin{aligned} \Delta y &\approx f'(a)\,h \\ f(a+h) - f(a) &\approx f'(a)\,h \\ f(a+h) &\approx f(a) + f'(a)\,h \end{aligned}$$

1. Suppose $f(x) = x^2$. Evaluate $f'(1)$. Use the Microscope Approximation to approximate $f(x)$ at $x = 1.02 = 1 + .02$. (Hint: What is Δx, or, if you prefer, h?)

2. What is your error in this example using this approximation? Do you think you would have a larger or smaller error if instead you approximated $f(1.002)$ using this method?

If we actually know the true value of $f(a+h)$, we can calculate the error incurred by using the Microscope Approximation to approximate it. However, even if we don't know the true value we can still write down an expression for the approximation error which will tell us some useful things. Since the function f and the value a are fixed, the error E can only depend on h.

$$\text{Error, } E(h) := \text{True Value} - \text{Approximate Value} = f(a+h) - \Big(f(a) + f'(a)\,h \Big).$$

NOTE: In this expression, the notation $E(h)$ does not mean "error times h". Rather, it is stating, in the usual function notation, that the error E *depends on* h.

3. As $h \to 0$, what happens to $E(h) = f(a+h) - \Big(f(a) + f'(a)\, h \Big)$? (Use the fact that since f is differentiable at a, it must also be continuous at a. Why?)

$$\lim_{h \to 0} E(h) =$$

Now notice that, from the definition of $E(h)$, we can write (and rewrite the expression for the *relative error* $E(h)/h$ as follows:

$$\frac{E(h)}{h} = \frac{f(a+h) - (\, f(a) + f'(a)\, h\,)}{h} = \left(\frac{f(a+h) - f(a)}{h} \right) - \frac{f'(a)\, h}{h}$$
$$= \left(\frac{f(a+h) - f(a)}{h} \right) - f'(a).$$

4. With this observation, what happens to the *relative error* $E(h)/h$ as $h \to 0$?

$$\lim_{h \to 0} \frac{E(h)}{h} =$$

We can summarize these results in the following Theorem.

Taylor's Theorem

Suppose f is differentiable at a. Then

$$f(a+h) = f(a) + f'(a)\, h + E(h),$$

where the Microscope Approximation error $E(h)$ satisfies

$$\lim_{h \to 0} E(h) = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{E(h)}{h} = 0.$$

We know that $f(a+h) \approx f(a) + f'(a)\, h + E(h)$ for any locally linear function. So it makes sense that to make the equation exact, we need to add a correction term. How much do we need to correct by? The error! So, Taylor's Theorem allows us to write simpler exact expressions for nonlinear functions. These simpler expressions have the form "linear function + error." The "linear function" part is called the **linearization** of the function about the specified point. We have seen these in our tangent line approximations, Euler's Method, and the First Degree Taylor Polynomial (recognize the name?). Such expressions are useful because algebra with linear functions is comparatively easy.

Example

Use Taylor's Theorem to write an "exact" expression for $\cos(\pi + h)$ in the form, "the linearization of $\cos x$ about π, plus error:"

$$\begin{aligned}
\cos(\pi + h) &= \cos(\pi) + \cos'(\pi)\, h + E(h) \\
&= \cos(\pi) - \sin(\pi)\, h + E(h), \qquad \text{since the derivative of } \cos x \text{ is } -\sin x, \\
&= -1 - 0 \cdot h + E(h) \\
&= -1 + E(h), \qquad \text{where } \lim_{h \to 0} E(h) = 0 \text{ and } \lim_{h \to 0} \frac{E(h)}{h} = 0.
\end{aligned}$$

5. Use Taylor's Theorem to write an exact expression for 2^{3+h} in the form, "the linearization of 2^x about 3, plus error:"

6. Write an expression for $E(h)$ in (5.). Plot it using your calculator. What do you see?