Qualitiative Analysis of the S-I-R Model

Qualitative analysis of a rate equation model means determining important features of its solution without estimating the entire solution. In this rest of this class we will apply this sort of analysis to the S-I-R model of a measles-like disease that we have been studying this week.

$$\begin{array}{ll} S(0) = S_0 & S'(t) = -a\,S(t)I(t) \\ I(0) = I_0 \ , & I'(t) = a\,S(t)I(t) - b\,I(t) \quad , \quad t \geq 0 \\ R(0) = R_0 & R'(t) = b\,I(t) \end{array}$$

where

$$\left.\begin{array}{rl} S(t), & \text{size of the } \textit{susceptible} \text{ population at time } t \\ I(t), & \text{size of the } \textit{infected} \text{ population at time } t \\ R(t), & \text{size of the } \textit{recovered} \text{ population at time } t \\ a, & \text{transmission coefficient} \\ b = 1/k, & \text{recovery coefficient} \\ k, & \text{average number of days it takes to recover from the illness.} \end{array}\right. \quad , \quad t \geq 0\,.$$

The Threshhold Value

An epidemic is said to occur if $I(t)$ rises to a peak, then dies away. We will start our investigation of the conditions under which an epidemic will occur by asking the question,

> "At the time the greatest number of people are infected,
> how many *susceptible* people are there?"

You may think that we would need to solve this model for the functions $S(t)$ and $I(t)$ in order to answer this question. Remarkably, however, this isn't necessary.

1. What is the "slope" of the graph of $I(t)$ when $I(t)$ reaches its maximum value?

2. What is the value of $I'(t)$ when $I(t)$ reaches its maximum value?

We will now define some new symbols:

I_*, the maximum value of $I(t)$

S_*, the value of $S(t)$ at the time when $I(t)$ reaches its maximum value

3. Use your answers to 1. and 2. above, together with the rate equation

$$I'(t) = a\,S(t)I(t) - b\,I(t)$$

to find a relationship between I_* and S_*.

4. Try to find an expression for S_* which does not depend on I_*.

The value you just found is called the *threshhold value* of $S(t)$:

$$S_* = b/a, \qquad \text{threshhold value}$$

To see why, study the graphs for $S(t)$, $I(t)$, and $R(t)$ you predicted when we began to study this model.

6. What happens to $I(t)$ *before* it reaches is maximum value?

7. What happens to $I(t)$ *before* $S(t)$ decreases all the way to the value S_*?

8. Suppose that $S(0)$, the *initial value* of $S(t)$, were *greater than* S_*. Sketch graphs of $I(t)$ and $S(t)$ in this case.

9. What happens to $I(t)$ *after* it reaches is maximum value?

10. What happens to $I(t)$ *after* $S(t)$ reaches the value S_*?

11. Suppose that $S(0)$, the *initial value* of $S(t)$, were *less than or equal to* S_*. Sketch graphs of $I(t)$ and $S(t)$ in this case.

An epidemic is said to occur if $I(t)$ rises to a peak, then dies away.
If $S(0) > S_*$, the threshhold value of S, then an epidemic will occur.
If $S(0) \leq S_*$, then an epidemic will not occur.

Can An Epidemic Be Avoided?

Suppose you are a public health official for a school district with 50,000 students. A measles outbreak is reported in a neighboring school district and you fear yours is next. You know that the average recovery time for a case of measles is 14 days, and that about 2.5% of contacts between an infected and a susceptible individual lead to infection. Suppose you estimate that on a typical day, an average individual has a probability of 2/5000 of coming in contact with an infected individual. If 45,400 of your district's students are initially susceptible, will an epidemic occur? Based on our analysis of the S-I-R model, can you recommend some measures which would prevent an epidemic in your district?

Asymptotic Analysis

Consider the generic S-I-R model with parameters a, b and c. What happens after the epidemic has been running through the population for a very very long time? In other words, what happens as $t \to \infty$? What is $S_\infty = \lim_{t\to\infty} S$? $I_\infty = \lim_{t\to\infty} I$? $R_\infty = \lim_{t\to\infty} R$?

[a.] Show that $(I + S - (b/a) \cdot \ln S)' = 0$

[b.] Therefore show that $S_\infty - \dfrac{b}{a} \ln(S_\infty) = I_0 + S_0 - \dfrac{b}{a} \ln(S_0)$

[c.] What do these results above tell us about the possibility of someone never getting the disease? In other words, what is the value of S_∞?