Successive Approximation and Euler's Method

Why Are Small Stepsizes Good?

As you know, when you apply Euler's Method to an initial value problem using a particular stepsize you get one piecewise linear function. This piecewise linear function is regarded as an approximation to the solution of the initial value problem.

Suppose you then apply Euler's Method to the same initial value problem using a smaller stepsize. This produces another piecewise linear function, interpreted as another approximation to the solution of the initial value problem.

1. From your experience with Euler's Method in this course, what evidence do you have that the approximation produced using the smaller stepsize is generally the better approximation?

2. From your understanding of Euler's Method, WHY do you think that using a smaller stepsize produces a better approximation?

Successive Approximations Are Better Than A Single Approximation

You might think that all you need to do with Euler's Method is pick a really small stepsize and use it to produce one piecewise linear approximation to the solution of the initial value problem. However, it turns out that it is better to repeat Euler's Method several times, each time using a smaller stepsize, to create a sequence of *successive* piecewise linear approximations. Here are some examples to illustrate why this is so.

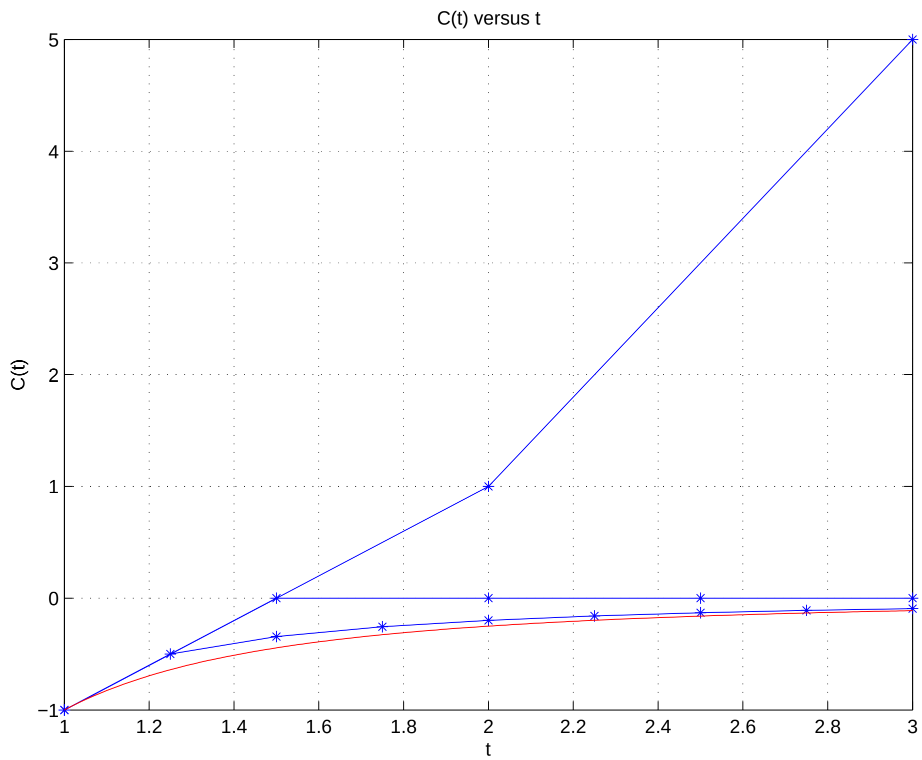
You Might Be Missing Something

Consider the following initial value problem:

$$C'(t) = 2t \cdot (C(t))^2, \qquad C(1) = -1.$$

On the plot below, piecewise linear approximations are generated using Euler's Method on the interval $1 \le t \le 3$ for the step sizes $\Delta t = 1$, $\Delta t = 1/2$ and $\Delta t = 1/4$.

The graph of the actual solution, $C(t) = -1/t^2$, is also plotted. (Remember, you'll learn the technique for finding the exact solution to initial value problems like this one later.)



3. What important feature of the true solution does the approximation using $\Delta t = 1$ fail to capture?

   The problem here is not the size of the original Δt per se, but the size of that Δt relative to how quickly the slope of the true solution changes. Thus, in a problem where the slopes changed very quickly, what might seem to be a small Δt might not be small enough.

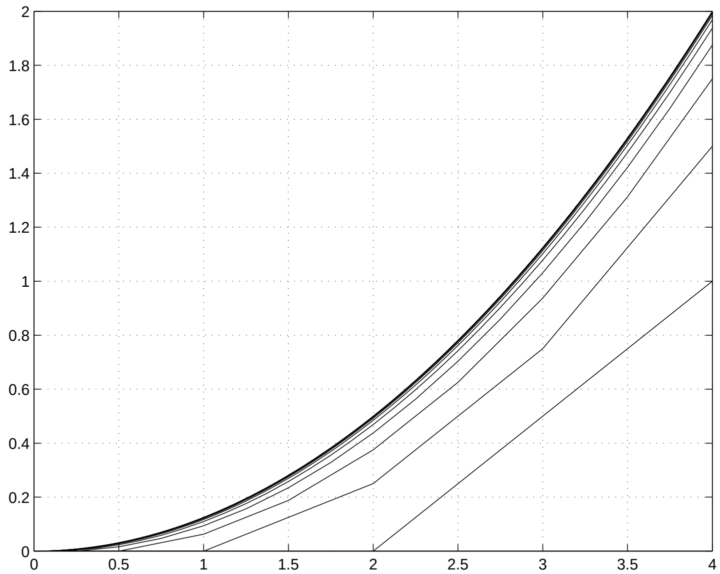
How Small Is Small Enough?

In practice, a sequence of Euler's Method approximations is produced, each with a smaller stepsize, until you notice very little change in approximations when the stepsize is further reduced. An example is given below for an initial value problem with which you are familiar:

$$y'(t) = \frac{1}{4}t, \qquad y(0) = 0.$$

The exact solution of this initial value problem is $y(t) = \frac{1}{8}t^2$. (Again, there are techniques for finding these solutions; the solutions aren't here by magic! These are just techniques you will learn later.)

In the plot below, you see what appear to be seven function graphs. Actually, there are TEN plotted there. This is a sequence of ten piecewise linear approximations to the solution, each one produced using Euler's Method using a stepsize half as large as before. The last three of these approximations are, however, essentially indistiguishable at this level of resolution. This gives one a lot of confidence that they are quite close to the true solution.



4. What does the *sequence* of approximations tell you in this case that no single one of them can tell you?

Successive Approximation with Euler's Method

The process of generating a *sequence* of approximations to the same thing is called *successive approximation*. If we run Euler's Method with one step size, we get one piecewise linear approximation to the solution of the initial value problem. If we run it again with another (generally smaller) step size, we get *another* piecewise linear approximation. If we run it yet again with another (generally smaller) step size, we get yet *another* approximation. In this way we can generate a *sequence* $Y_1(t), Y_2(t), Y_3(t)\ldots$ of piecewise-linear approximations to the solution $y(t)$.

The idea we have been building up to is that the sequence of piecewise-linear functions produced by Euler's Method is, in some sense, *converging* to a function. This function is the solution of the initial value problem used in Euler's method. With this idea – that we will be getting closer and closer to a single solution – we need to start discussing the quality of Euler approximations, how the quality relates to stepsize, and how we can sometimes use an understanding of this relationship to predict the actual solution.

With a sequence of approximations, we can see patterns which are not evident in a single approximation. These patterns can give you important information about the quality of your approximations.

5. Suppose you have two variables A and B, and you know that the ratio A/B is constant, i.e. $A/B = k$ or $A = kB$.

   Complete the following statement and explain your answer: "A is   to B."

   Euler Error at a Point is Approximately Proportional to Stepsize

   We now shift our focus from the entire approximation produced by Euler's Method to an individual point. For example, it can be shown (Math 120 again) that the correct solution to the initial value problem

$$y'(t) = 3t^2, \qquad y(0) = 0$$

   has the value 8 when $t = 2$. That is $y(2) = 8$.

   What does Euler's Method predict that the value of y will be at $t = 2$? It depends on the stepsize, of course. The table below shows the results of Euler's Method for a sequence of decreasing stepsizes. However, rather than tabulating the estimates for $y(2)$, this table give the values of the ratio (estimated $y(2) - 8$)$/\Delta t$.

Δt	$\frac{(\text{estimated } y(2)-8)}{\Delta t}$
2^1	-4
2^0	-5
2^{-1}	-5.5
2^{-2}	-5.75
2^{-3}	-5.875
2^{-4}	-5.9375
2^{-5}	-5.96875
2^{-6}	-5.984375
2^{-7}	-5.9921875
2^{-8}	-5.99609375
2^{-9}	-5.998046875

6. What does this table tell you about the relationship between the estimation error and the stepsize as the stepsize gets closer and closer to zero? Why can't you necessarily discover this relationship by looking at only one approximation?