HOMEWORK 5 SOLUTIONS:

Set 1: SM 2.2: 5–7, 24, 25

Set 2: SM 2.2: 18, 56, 57; 2.5: 6, 10

Set 3: Worksheet (attached)

BONUS: SM 2.5: 53

Set 1: SM 2.2: 5–7, 24, 25

5. $f(x) = 3x + 1$, $a = 1$

Two ways:

(a) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a+h) + 1 - (3a+1)}{h}$$

$$= \lim_{h \to 0} \frac{3a + 3h + 1 - 3a - 1}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

$$\boxed{f'(a) = 3 \Rightarrow f'(1) = 3}$$

OR

(b) $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h) + 1 - (3 \cdot 1 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} + 3h \cancel{+1} - \cancel{3} \cancel{-1}}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

$$\boxed{f'(1) = 3}$$

6. $f(x) = 3x^2 + 1$, $a = 1$

Two ways:

(a) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a+h)^2 + 1 - (3a^2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{3(a^2 + 2ah + h^2) + 1 - 3a^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 + 1 - 3a^2 - 1}{h}$$

cont'd next page

6 cont'd

$$= \lim_{h\to 0} \frac{6ah+3h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(6a+3h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6a+3h = 6a$$

$$\boxed{f'(a)=6a \Rightarrow f'(1)=6}$$

OR

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{3(1+h)^2+1-(3(1)^2+1)}{h}$$

$$= \lim_{h\to 0} \frac{3(1+2h+h^2)+1-3-1}{h}$$

$$= \lim_{h\to 0} \frac{3+6h+3h^2-3}{h} = \lim_{h\to 0} \frac{6h+3h^2}{h}$$

$$= \lim_{h\to 0} 6+3h = 6 \qquad \boxed{f'(1)=6}$$

7. $f(x)=\sqrt{3x+1}$; $a=1$

Two ways

(a) $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{3(a+h)+1}-\sqrt{3a+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3a+3h+1}-\sqrt{3a+1}}{h} \cdot \frac{\sqrt{3a+3h+1}+\sqrt{3a+1}}{\sqrt{3a+3h+1}+\sqrt{3a+1}}$$

$$= \lim_{h\to 0} \frac{(\cancel{3a}+3h+\cancel{1})-\cancel{(3a+1)}}{h[\sqrt{3a+3h+1}+\sqrt{3a+1}]}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}[\sqrt{3a+3h+1}+\sqrt{3a+1}]} = \frac{3}{\sqrt{3a+0+1}+\sqrt{3a+1}} = \frac{3}{2\sqrt{3a+}}$$

$$\boxed{f'(a)=\frac{3}{2\sqrt{3a+1}} \Rightarrow f'(1)=\frac{3}{2\sqrt{3+1}} = \frac{3}{4}}$$

57. $\ulcorner \to \lrcorner$ in 180°

This should take $\approx 180\cancel{^\circ} \cdot \frac{\cancel{ms}}{11\cancel{^\circ}} \cdot \frac{.001\,s}{1\,\cancel{ms}} \approx \boxed{.016\,s}$

$\urcorner \to \lrcorner$ in 90°

This should take $\approx 90^\circ \cdot \frac{ms}{11^\circ} \cdot \frac{.001\,s}{1\,ms} \approx \boxed{.008\,s}$

2.5: 6, 10

6. $f(x) = x^2 + 2\cos x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 2\cos(x+h) - (x^2 + 2\cos x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + 2(\cos x \cosh - \sin x \sin h) \cancel{- x^2} - 2\cos x}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 2\cos x \cos h - 2\cos x - 2\sin x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}} + \lim_{h\to 0} 2\cos x\left(\frac{\cos h - 1}{h}\right) - \lim_{h\to 0} \frac{2\sin x \sin h}{h}$$

$$= \lim_{h\to 0} (2x+h) + \lim_{h\to 0} 2\cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \lim_{h\to 0} 2\sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= 2x + 2\cos x \cdot 0 - 2\sin x \cdot 1 = 2x - 2\sin x$$

$$\boxed{f'(x) = 2x - 2\sin x}$$

10. $f(x) = 4x^2 - 3\tan x$

$$f'(x) = \lim_{h\to 0} \frac{4(x+h)^2 - 3\tan(x+h) - (4x^2 - 3\tan x)}{h}$$

$$= \lim_{h\to 0} \frac{4(x^2 + 2xh + h^2) - 3\tan(x+h) - 4x^2 + 3\tan x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4x^2} + 2xh + h^2 - \cancel{4x^2}}{h} + \lim_{h\to 0} \frac{3\tan x - 3\tan(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} + 3\lim_{h\to 0} \frac{\tan x - \tan(x+h)}{h}$$

$$= \lim_{h\to 0} 2x+h + 3\lim_{h\to 0} \frac{\frac{\sin x}{\cos x} - \frac{\sin(x+h)}{\cos(x+h)}}{h} \cdot \frac{\cos x\cos(x+h)}{\cos x\cos(x+h)}$$

$$= 2x + 3\lim_{h\to 0} \frac{\sin x\cos(x+h) - \cos x\sin(x+h)}{h\cos x\cos(x+h)}$$

$$= 2x + 3\lim_{h\to 0} \frac{\sin x(\cos x\cosh - \sin x\sinh) - \cos x(\sin x\cosh + \sinh\cos x)}{h\cos x\cos(x+h)}$$

$$= 2x + 3\lim_{h\to 0} \frac{\sin x\cancel{\cos x\cosh} - \sin^2 x\sinh - \sin x\cancel{\cos x\cosh} - \sinh\cos^2 x}{h\cos x\cos(x+h)}$$

$$= 2x + 3\lim_{h\to 0} \frac{-\sin^2 x\sinh - \sinh\cos^2 x}{h\cos x\cos(x+h)}$$

$$= 2x + 3\lim_{h\to 0} -\frac{\sin h}{h}\left(\frac{\sin^2 x + \cos^2 x}{\cos x\cos(x+h)}\right)$$

$$= 2x + -3\lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos x\cos(x+h)} = 2x - 3\cdot 1\cdot \sec^2 x$$

$$\boxed{f'(x) = 2x - 3\sec^2 x}$$

BONUS: SM 2.5: 53

We have

$$f(t) = \begin{cases} 1 & 26 \le t \le 30 \\ g(t) & 30 \le t \le 34 \\ 0 & 34 \le t \le 36 \end{cases}$$

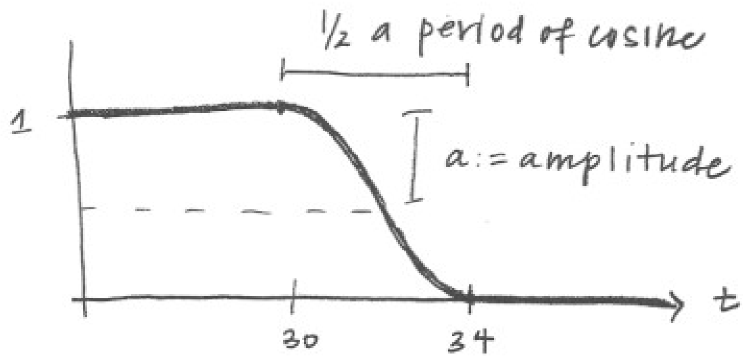
We want $g(t) = a\cos(bt) + c$ so that $f(t)$ is differentiable at $t=30$ and $t=34$.

This means $\lim_{h\to 0^-} \frac{f(30+h) - f(30)}{h} = g'(30)$

AND $\lim_{h\to 0^+} \frac{f(34+h) - f(34)}{h} = g'(34)$

AND $g(30) = f(30) = 1$, $g(34) = f(34) = 0$

$f(t)$ looks like:



From the graph (and properties of $\cos x$) we know

$a = \frac{1}{2}$ (amplitude)

$c = \frac{1}{2}$ (vertical shift)

We also know that half a period of $\cos x$ should be equal to 4 (34−30). If $f(t) = \cos(bt)$, the period is $\frac{2\pi}{b}$.

So we want $8 = \frac{2\pi}{b} \Rightarrow 8b = 2\pi \Rightarrow b = 2\pi/8 = \pi/4$

But we also need a horizontal shift

This leaves us with: $g(t) = \frac{1}{2}\cos\left(\frac{\pi}{4}(t-30)\right) + \frac{1}{2}$

Let's check $g'(30)$ and $g'(34)$:

$$g'(t) = \frac{1}{2}\left(-\sin\left(\frac{\pi}{4}(t-30)\right)\right)\cdot\frac{\pi}{4} + 0$$

$$= -\frac{\pi}{8}\sin\left(\frac{\pi}{4}(t-30)\right)$$

$$g'(30) = -\frac{\pi}{8}\sin(0) = 0 \checkmark \quad 0 = \lim_{h\to 0^-}\frac{f(30+h) - f(3}{h}$$

$$g'(34) = \frac{-\pi}{8}\sin\left(\frac{\pi}{4}(34-30)\right)$$

$$= \frac{-\pi}{8}\sin(\pi) = 0 \checkmark \quad 0 = \lim_{h\to 0^+}\frac{f(34+h) - f(}{h}$$

So $g(t) = \frac{1}{2}\cos\left(\frac{\pi}{4}(t-30)\right) + \frac{1}{2}$ is the function that allows $f(t)$ to be continuous and differentiable at $t =$ and $t = 34$, since $g'(30) = 0 = \lim_{h\to 0^-}\frac{f(30+h) - f(30)}{h}$ and

$g'(34) = 0 = \lim_{h\to 0^+} \frac{f(34+h) - f(34)}{}$.

Taylor's Theorem and Error in Tangent Line Approximations

Name:

1. In class we saw the following equation for the error in a tangent line approximation.

$$\text{Error}, E(h) = \text{True Value} - \text{Approximate Value} = f(a+h) - (f(a) + f'(a)h).$$

(a) Explain, using complete sentences, why we write $E(h)$. Be sure to state what both E and h are. (*This is related to what the function notation means.*)
(b) Explain, in complete sentences, how the approximation error changes as we get closer and farther from the value a.

(a) We write $E(h)$ since the error is a function of h, i.e. as h changes, so will the error.

(b) The approximation error decreases as h decreases so that we are closer to a; similarly as h increases and we are approximating values farther from a, the approximation error increases.

2. For the following functions, find the equation of the tangent line at $x = 0$ and find a formula for the error $E(h)$. Then approximate $f(1)$ and evaluate the error in the approximation.

(a) $f(x) = \dfrac{1}{x+3}$

tangent line: $y = f(a) + f'(a)(\,)$

$f(0) = \frac{1}{3}$

$f'(x) = \dfrac{-1}{(x+3)^2}$

$f'(0) = \dfrac{-1}{3^2} = -\dfrac{1}{9}$

tangent line at $x=0$:

$y = \frac{1}{3} + -\frac{1}{9}(x-0) = \frac{1}{3} - \frac{1}{9}x$

$f(1) \approx \frac{1}{3} - \frac{1}{9}(1) = \frac{3}{9} - \frac{1}{9} = \frac{2}{9}$

$f(1) - \tilde{f}(1) = \frac{1}{1+3} - \frac{2}{9} = \frac{1}{4} - \frac{2}{9} = \boxed{\frac{1}{36} := \text{error}}$

(b) $f(t) = t^2 + 3t + 5$

$f(0) = 5$

$f(1) = 1+3+5 = 9$

$f'(t) = 2t + 3$

$f'(0) = 3$

tangent line at $x=0$:

$y = 5 + 3(x-0) = 5 + 3x$

$f(1) \approx 5 + 3(1) = 8$

$f(1) - \tilde{f}(1) = 9 - 8 = \boxed{1 := \text{error}}$

(c) $f(x) = xe^x$

$f(0) = 0 \cdot e^0 = 0$

$f(1) = 1 \cdot e^1 = e$

$f'(x) = xe^x + e^x$

$f'(0) = 0e^0 + e^0 = 1$

tangent line at $x=0$:

$y = 0 + 1(x-0) = x$

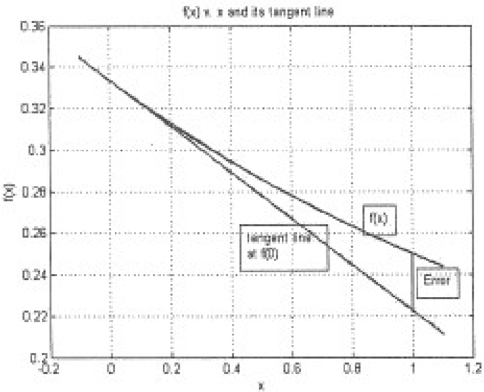
$f(1) \approx 1$

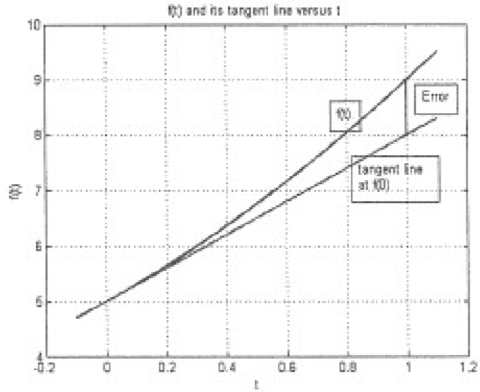
$f(1) - \tilde{f}(1) = \boxed{e - 1 := \text{error}}$

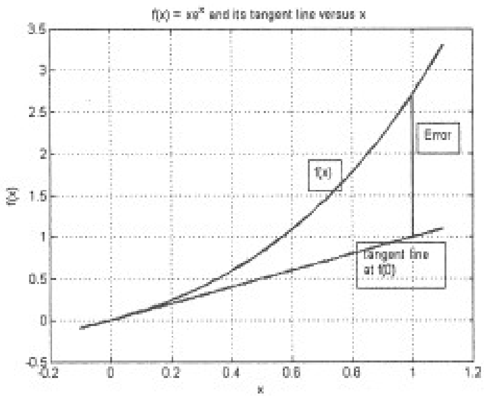
Taylor's Theorem and Error in Tangent Line Approximations

3. Pick one of the functions in *(2.)*, graph the function and the tangent line and illustrate the error in your approximation to $f(1)$.

The following figures show the graphs for each function in part *(2.)*. They are only graphed over the interval $[-0.1, 1.1]$.







4. Pick one of the functions in *(2.)* along with its error formula $E(h)$ and show that

$$\lim_{h\to 0} E(h) = 0; \qquad \text{and} \qquad \lim_{h\to 0} \frac{E(h)}{h} = 0$$

for (a)

$$E(h) = f(a+h) - (f(a) + f'(a)h) \qquad a=0$$

$$= \frac{1}{(0+h)+3} - \left(\frac{1}{3} - \frac{1}{9}h\right)$$

$$= \frac{1}{h+3} - \frac{1}{3} + \frac{1}{9}h \qquad E(h) = \frac{1}{h+3} + \frac{h}{9} - \frac{1}{3}$$

$$\lim_{h\to 0} E(h) = \lim_{h\to 0} \frac{1}{h+3} + \frac{h}{9} - \frac{1}{3} = \frac{1}{3} + 0 - \frac{1}{3} = 0 \checkmark$$

$$\lim_{h\to 0} \frac{E(h)}{h} = \lim_{h\to 0} \frac{\frac{1}{h+3} + \frac{h}{9} - \frac{1}{3}}{h} \cdot \frac{(9(h+3))}{(9(h+3))}$$

$$= \lim_{h\to 0} \frac{9 + h(h+3) - 3(h+3)}{h \cdot 9 \cdot (h+3)}$$

$$= \lim_{h\to 0} \frac{\cancel{9} + h^2 + \cancel{3h} - \cancel{3h} - \cancel{9}}{9h(h+3)}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}}}{9\cancel{h}(h+3)} = \lim_{h\to 0} \frac{h}{9(h+3)} = \frac{0}{27} = 0 \checkmark$$

for (b)

$$E(h) = f(a+h) - (f(a) + f'(a)h) \qquad a=0$$

$$= (0+h)^2 + 3(0+h) + 5 - (5+3h)$$

$$= h^2 + 3h + 5 - 5 - 3h = h^2 \qquad E(h) = h^2$$

$$\lim_{h\to 0} E(h) = \lim_{h\to 0} h^2 = 0 \quad \checkmark$$

$$\lim_{h\to 0} \frac{E(h)}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0 \quad \checkmark$$

for (c):

$$E(h) = f(a+h) - (f(a) + f'(a)h) \qquad a=0$$

$$= (0+h)e^{(0+h)} - (0 + 1\cdot h)$$

$$= he^h - h = h(e^h - 1) \qquad E(h) = h(e^h - 1)$$

$$\lim_{h\to 0} E(h) = \lim_{h\to 0} h(e^h - 1) = 0(1-1) = 0 \quad \checkmark$$

$$\lim_{h\to 0} \frac{E(h)}{h} = \lim_{h\to 0} \frac{\cancel{h}(e^h - 1)}{\cancel{h}} = \lim_{h\to 0} e^h - 1 = 1 - 1 = 0 \quad \checkmark$$