Homework 12 Solutions: SM 3.7: 6, 23, 26, 31, 32, 33, 41

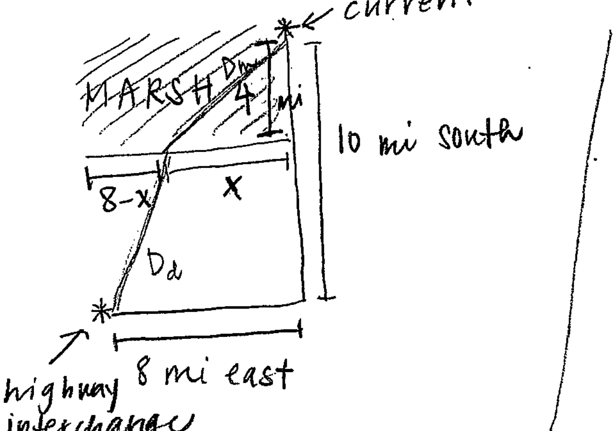
3.6: 31; Chapter 3 Review: 46

BONUS: Chapter 3 Review # 28

3.7: 6, 23, 26, 31, 32, 33, 41

6. This statement is true because e^x is an increasing function of x; i.e. as x increases so does e^x. Thus $e^{f(x)}$ will be minimized when $f(x)$ is minimized.

23. picture:



We want to minimize cost. The cost of the highway over the marsh is $\frac{5 \text{ million}}{\text{mile}} \cdot$ distance over marsh

and over the other land is $\frac{2 \text{ million}}{\text{mi}} \cdot D$ ($D :=$ distance). The distance over the marsh land is given by $D_m = \sqrt{4^2 + x^2}$ where x is the distance east from the current bridge. The distance over dry land is $D_d = \sqrt{6^2 + (8-x)^2}$. So the cost function to minimize is

$$C(x) = 5\sqrt{16 + x^2} + 2\sqrt{100 - 16x + x^2} \qquad (C(x) \text{ is in millions})$$

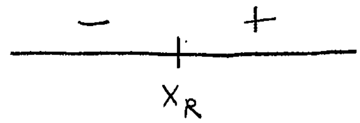
$$\text{So: } C'(x) = \frac{5}{\cancel{2}}(16 + x^2)^{-1/2}(\cancel{2}x) + \frac{\cancel{2}}{\cancel{2}}(100 - 16x + x^2)^{-1/2}(-16 + 2x)$$

$$C'(x) = \frac{5x}{\sqrt{16 + x^2}} + \frac{2x - 16}{\sqrt{100 - 16x + x^2}}$$

Set this equal to zero and solve for critical values. Note that the domain for x is $0 \le x \le 8$ (and it is closed). Use Newton's method or some computing mechanism to solve for the root:

$$x_R \approx 1.2529$$

check the sign of $C'(x)$ on either side



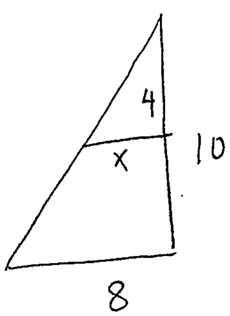
So at x_R is a local minimum.

Now compare $C(1.2529)$ to $C(0)$ and $C(8)$:

$C(1.2529) \approx 39$ mil. $C(0) = 40$ mil. $C(8) \approx 56.7$ mil.

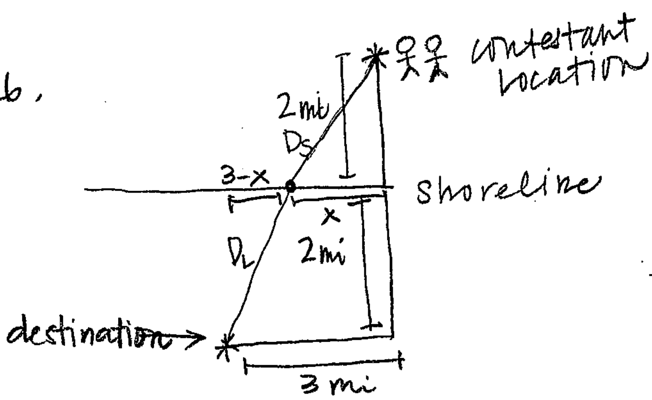
(a) The highway should emerge from the marsh ≈ 1.2529 miles east of the current bridge to minimize the highway cost.

(b)



If the highway is built in a straight line (see figure at left) then x is 3.2 miles east of the current bridge ($\frac{4}{x} = \frac{10}{8}$ by similar triangles) Then $C(3.2) \approx 41$ million. So almost 2 million dollars are saved.

26.



We want to minimize time. Note that time = distance/speed. Total time (in hours) is given by:

$$T(x) = \frac{\sqrt{4+x^2}}{4} + \frac{\sqrt{4+(3-x)^2}}{10}$$

where x is the distance (in miles) east of the current position.

Following the same procedure.

$$T'(x) = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{\cancel{2}}\right)(4+x^2)^{-1/2}(\cancel{2}x) + \tfrac{1}{10}\left(\tfrac{1}{\cancel{2}}\right)(13-6x+x^2)^{-1/2}(\overset{-3+x}{\cancel{-6+2x}})$$

$$= \frac{x}{4\sqrt{4+x^2}} + \frac{x-3}{10\sqrt{13-6x+x^2}}$$

$x_R \approx 0.6407$ signs of derivative $\underset{x_R}{\overset{- \qquad +}{\longrightarrow}}$

The domain of x is $0 \le x \le 3$ (it is closed!)

at x_R there is a local minimum, we compare $T(0), T(3), T(x_R)$.

$T(0) \approx .86$ hours ; $T(x_R) \approx .83$ hours; $T(3) \approx 1.1$ hours.

(a) The contestant should swim to the shoreline $\approx .6407$ miles east of their current position to minimize their total time.

(b) If emerging on the shoreline $\approx .6407$ miles east of their current position, contestants will spend $\approx .525$ hours in the water (31.5 minutes) and $\approx .309$ hours on land (18.5 minutes).

31. $V(r) = cr^2(r_0 - r)$ $\quad$ c and r_0 are constants.

$V'(r) = cr^2(-1) + 2cr(r_0 - r)$ $\quad$ (product rule)

$= -cr^2 + 2cr_0 r - 2cr^2$

$V'(r) = 2cr_0 r - 3cr^2$

$V'(r_*) = 0 = 2cr_0 r - 3cr^2 = cr(2r_0 - 3r)$

$\Rightarrow r_* = 0$ or $r_* = \frac{2}{3} r_0$

$V''(r) = 2cr_0 - 6cr$

$V''(0) = 2cr_0 > 0 \Rightarrow$ Local min @ $r_* = 0$

$V''(\frac{2}{3}r_0) = 2cr_0 - 6c(\frac{2}{3}r_0) = 2cr_0 - 4cr_0 < 0 \Rightarrow$ Local max @ $r_* = \frac{2}{3} r_0$.

$\frac{2}{3}r_0$ is the only candidate for a local max so it is our maximum value.

$\boxed{\text{At } \frac{2}{3}r_0 \text{ the maximum velocity occurs; this implies that the windpipe contracts.}}$

32. $E(\theta) = \dfrac{\csc\theta}{r^4} + \dfrac{1-\cot\theta}{R^4}$ (r, R are constants)

$$\frac{dE}{d\theta} = \frac{1}{r^4}(-\csc\theta\cot\theta) + \frac{1}{R^4}(-\csc^2\theta)$$

$$= -\csc\theta\left(\frac{\cot\theta}{r^4} - \frac{\csc\theta}{R^4}\right)$$

$$\frac{dE}{d\theta} = 0 = -\csc\theta\left(\frac{\cot\theta}{r^4} - \frac{\csc\theta}{R^4}\right)$$

either $-\csc\theta = 0$ or $\dfrac{\cot\theta}{r^4} - \dfrac{\csc\theta}{R^4} = 0$

$-\dfrac{1}{\sin\theta} = 0$ $\qquad \dfrac{\cot\theta}{r^4} = \dfrac{\csc\theta}{R^4}$

not possible.

$$\frac{\cot\theta}{\csc\theta} = \frac{r^4}{R^4} = \frac{\cos\theta/\sin\theta}{1/\sin\theta} = \cos\theta$$

So, $\cos\theta = r^4/R^4$ or $\theta = \cos^{-1}(r^4/R^4)$

$\uparrow$

possible since $r<R$ and so $r^4/R^4 < 1$.

We only have one candidate:

$\boxed{\text{The angle } \theta = \cos^{-1}(r^4/R^4) \text{ minimizes energy loss.}}$

33. $p(x) = \dfrac{V^2 x}{(R+x)^2}$ V and R are constants/parameters.

$$\frac{dp}{dx} = \frac{(R+x)^2(V^2) - (V^2x)(2)(R+x)(1)}{(R+x)^4}$$ quotient rule.

$$\frac{dp}{dx} = \frac{V^2(R+x)^2 - 2V^2x\,(R+x)}{(R+x)^4} = \frac{V^2\cancel{(R+x)}\left[(R+x) - 2x\right]}{(R+x)^{\cancel{4}}}$$

$$\frac{dp}{dx} = \frac{V^2(R-x)}{(R+x)^3} \qquad \frac{dp}{dx} = 0 = \frac{V^2(R-x)}{(R+x)^3} \Rightarrow x = R$$

check derivative signs $+$ | $-$ at $x=R$

So at $x=R$ there is a local maximum — it is the only candidate so it is the location of the maximum power absorption:

At $x=R$ the maximum amount of power is absorbed.
(The max amount is $\frac{V^2R}{(R+R)^2} = \frac{V^2}{4R}$ power units.)

41. $R = \dfrac{2v^2\cos^2\theta}{g}(\tan\theta - \tan\beta)$

$$\frac{dR}{d\theta} = \frac{2v^2\cos^2\theta}{g}(\sec^2\theta) + \frac{2v^2}{g}2\cos\theta(-\sin\theta)(\tan\theta - \tan\beta)$$
(product rule)

$$= \frac{2v^2\cos^2\theta}{g}\left(\frac{1}{\cos^2\theta}\right) + \frac{-4v^2}{g}\cos\theta\sin\theta\left(\frac{\sin\theta}{\cos\theta} - \tan\beta\right)$$

$$\frac{dR}{d\theta} = \frac{2v^2}{g} - \frac{4v^2}{g}\sin^2\theta + \frac{2v^2}{g}(2\cos\theta\sin\theta)\tan\beta$$

$$= \frac{2v^2}{g} - \frac{4v^2}{g}\sin^2\theta + \frac{2v^2}{g}\sin 2\theta\tan\beta$$

$$= \left(\frac{2v^2}{g}\right)\left[1 - 2\sin^2\theta + \sin 2\theta\tan\beta\right]$$

$$= \frac{2v^2}{g}\left[\sin^2\theta + \cos^2\theta - 2\sin^2\theta + \sin 2\theta\tan\beta\right]$$

$$= \frac{2v^2}{g}\left[(\cos^2\theta - \sin^2\theta) + \sin 2\theta\tan\beta\right]$$

$$= \frac{2v^2}{g}\left[\cos 2\theta + \sin 2\theta\tan\beta\right]$$

$$\frac{dR}{d\theta} = 0 = \frac{2v^2}{g}\left[\cos 2\theta + \sin 2\theta\tan\beta\right]$$

$$\Rightarrow \cancel{\frac{2v^2}{g}}\cos 2\theta = -\cancel{\frac{2v^2}{g}}\sin 2\theta\tan\beta$$

$$-\cot 2\theta = \tan\beta$$

$$-\cot 2\theta = -\tan\left(\frac{\pi}{2} - 2\theta\right) = \tan(2\theta - \pi/2)$$

$$\Rightarrow \tan(2\theta - \pi/2) = \tan\beta$$

so $\beta = 2\theta - \pi/2$

or $\boxed{\theta = \frac{\beta}{2} + \pi/4 \text{ will maximize } R}$ ✓

cont'd next page

(a) $\boxed{\beta = 10°,\ \theta = 50°}$

$$\theta = \frac{\beta}{2} + \pi/4 = 10°/2 + 45° = 50°$$

(b) $\boxed{\beta = 0°,\ \theta = 45°}$

$$\theta = \frac{0}{2} + 45$$

(c) $\boxed{\beta = -10°,\ \theta = 40°}$

$$\theta = -10°/2 + 45°$$

3.6 # 31. $f(x) = x + \sin x$

$f'(x) = 1 + \cos x$

$f'(x) = 0 \Rightarrow -1 = \cos x$

$\cos x = -1$ when $x = n\pi$ (n odd)

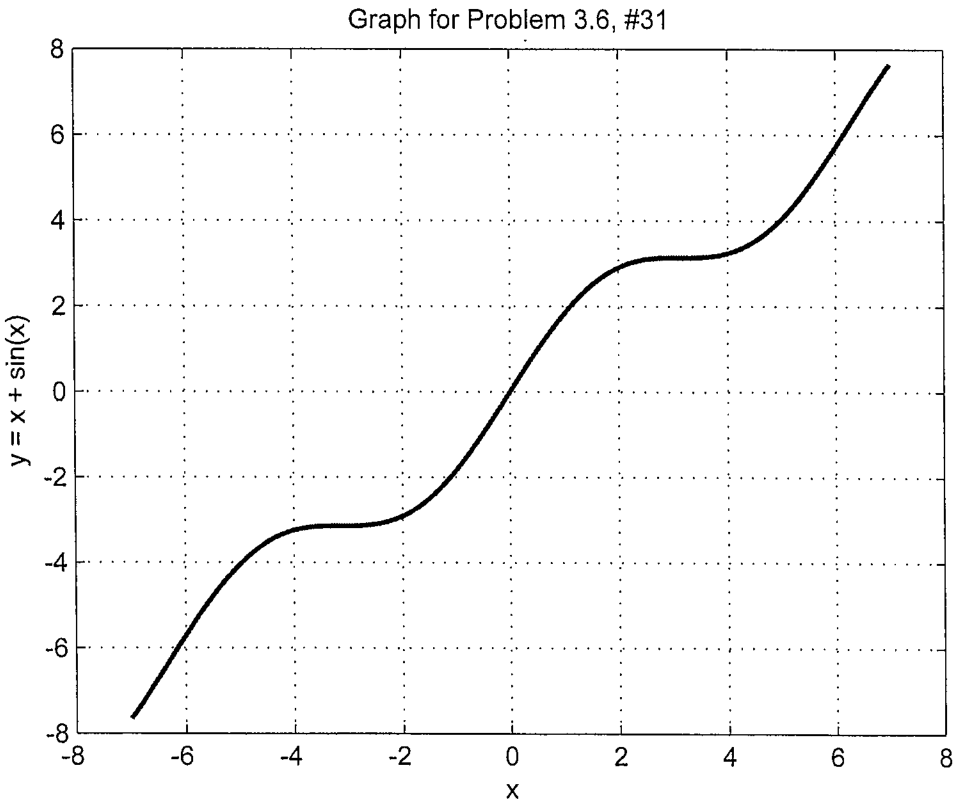
Note that $f'(x) \geq 0$ for all x, so f is increasing for all real numbers except when it equals 0 (at $x = n\pi$, n odd)

$f''(x) = -\sin x$, $f''(x) = 0 \Rightarrow \sin x = 0$ at $x = n\pi$ (n an integer)

$f''(x) > 0$ for $(n-1)\pi < x < n\pi$ (n even)

$f''(x) < 0$ for $n\pi < x < (n+1)\pi$ (n even)

And all values of $n\pi$ are points of inflection.


Graph for Problem 3.6, #31
y = x + sin(x)
x
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8

Chapter 3 Review # 4b:

$$x'(t) = .3\,x(t)\,[4 - x(t)]$$

This is the equation for the reaction rate. This is what we want to maximize. So we look at the 2nd derivative.

$$x''(t) = .3\,[4 - x(t)] + .3\,x(t)\,[-1]$$ product rule

$$= 1.2 - .3\,x(t) - .3\,x(t)$$

$$= 1.2 - .6\,x(t)$$

$$x''(t) = 0 \Rightarrow 1.2 - .6\,x(t) = 0$$

$$1.2 = .6\,x(t)$$

$$x(t) = 2$$

check signs of $x''(t)$:

$$\begin{array}{ccc} + & & - \\ \hline & 2 & \end{array}$$

$\boxed{\text{When } x(t) = 2 \text{ the maximum reaction rate occurs.}}$

BONUS: Chapter 3 Review # 28

$f'(x) > 0$ for all $x \neq 0$ (always increasing)

$f'(0)$ undefined

$f''(x) > 0 \quad x < 0$ (concave up)

$f''(x) < 0 \quad x > 0$ (concave down)

some examples: