37. The differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ can be rewritten as $\frac{dy}{dx} = \frac{1+(y/x)}{1-(y/x)} = f\left(\frac{y}{x}\right)$. In general, any differential equation that can be rewritten as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is called *homogeneous*. Show that every homogeneous differential equation can be reduced to a separable equation (in terms of x and v) by putting $y(x) = xv(x)$.

38. The differential equation $\frac{dt}{ds} = \frac{s+t+5}{s-t-1}$ is neither separable nor homogeneous (as defined in Problem 37). Show that making a change of variable of the form $s = x + a$, $t = y + b$, and choosing a and b appropriately, yields a homogeneous equation. The homogeneous equation can then be solved with the method of Problem 37.

).5 GROWTH AND DECAY

In this section we look at exponential growth and decay equations. Consider the population of a region. If there is no immigration or emigration, the rate at which the population is changing is often proportional to the population. In other words, the larger the population, the faster it is growing, because there are more people to have babies. If the population at time t is P, and its continuous growth rate is 2% per unit time, then we know

$$\text{Rate of growth of population} = 2\%(\text{Current population})$$

and we can write this as

$$\frac{dP}{dt} = 0.02P.$$

The 2% growth rate is called the *relative growth rate* to distinguish it from the absolute growth rate, or rate of change of the population, dP/dt. Notice they are in different units. Since

$$\text{Relative growth rate} = 2\% = \frac{1}{P}\frac{dP}{dt}$$

the relative growth rate is in percent per unit time, while

$$\text{Absolute growth rate} = \text{Rate of change of population} = \frac{dP}{dt}$$

is in people per unit time.

The equation $dP/dt = 0.02P$ is of the form $dP/dt = kP$ for $k = 0.02$ and therefore has the solution

$$P = P_0 e^{0.02t}.$$

Other processes are described by differential equations similar to that for population growth, but with negative values for k. In summary, we have the following result from the preceding section:

> Every solution to the equation
>
> $$\frac{dP}{dt} = kP$$
>
> can be written in the form
>
> $$P = P_0 e^{kt},$$
>
> where P_0 is the initial value of P, and $k > 0$ represents growth, while $k < 0$ represents decay.

Recall that the *doubling time* of an exponentially growing quantity is the time required for it to double. The *half-life* of an exponentially decaying quantity is the time for half of it to decay.

the object of mass m at an altitude h above the surface of the earth is given by

$$F = \frac{mgR^2}{(R+h)^2},$$

where R is the radius of the earth.

(a) Use Newton's Law of Motion to show that

$$\frac{dv}{dt} = -\frac{gR^2}{(R+h)^2}.$$

(b) Rewrite this equation with h instead of t as the independent variable using the chain rule $\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}$. Hence show that

$$v\frac{dv}{dh} = -\frac{gR^2}{(R+h)^2}.$$

(c) Solve the differential equation in part (b).
(d) Find the escape velocity, the value of v_0 such that v is never zero.

Problems 19–21 refer to the model of the expansion of the universe given on page 526 by the equations

$$R'' = -\frac{GM_0}{R^2} \quad \text{and} \quad (R')^2 = \frac{2GM_0}{R} + C,$$

where $R(t)$ is the radius of the universe (assumed spherical), t is time, G is the universal gravitational constant, and M_0 is the mass of the universe. (In the text we assumed that $C > 0$.)

19. In this problem we look at the case where $C < 0$. Writing $C = -K$, where $K > 0$, we have

$$R' = \pm\sqrt{\frac{2GM_0}{R} - K}.$$

Since the universe is currently expanding, at this time $R' > 0$, so R' is currently given by the positive root. Show that R increases to some value $R_{\max}$, and then decreases again. In addition, show that when this happens and R again approaches zero, then R' has a large negative value; in other words, a "big crunch" happens.

20. In this problem we look at the case where $C = 0$. Then we know that

$$R' = \sqrt{\frac{2GM_0}{R}}.$$

Solve this differential equation, assuming $R = 0$ when $t = 0$. Give R as a function of t. The resulting formula for R is called the "flat universe model." What does this model predict about the expansion of the universe?

21. (a) Einstein, who formulated the model of the universe described before Problem 19, wanted the universe to be stable—neither expanding nor shrinking. Why don't these differential equations allow for a stable universe?
(b) One estimate for the age of the universe is $R(t_0)/R'(t_0)$ where t_0 is the current time. (This number is called the *Hubble constant.*) Why is this a reasonable estimate for the age of the universe? Is it an overestimate or an underestimate?

10.7 MODELS OF POPULATION GROWTH

Population projections have been important to political philosophers since at least the late eighteenth century. As concern for scarce resources has grown, so has interest in accurate population projections. In this section we will look at two differential equations which are used to model both human and animal population growth. These differential equations have applications in economics and medicine, such as modeling the spread of an innovation or the growth of a tumor.

Relative versus Absolute Growth Rates

When describing population growth, we often use percentages rather than absolute numbers. For example, we say the population of the world is now about 5.8 billion people and growing at a continuous rate of about 1.7% a year, meaning that the relative growth rate is 1.7%:

$$\frac{dP/dt}{P} = \frac{1}{P}\frac{dP}{dt} = 0.017.$$

The quantity dP/dt is called the *absolute growth rate* and measures the growth rate in, say, pe per year. The quantity $(dP/dt)/P$ is called the *relative growth rate* and represents the absc growth rate as a fraction of the whole population. Its units are, say, % per year. When talking al populations we often use the relative growth rate.

The US Population: 1790--1860

Every ten years the population of the United States is recorded by a census. The first such ce was in 1790. Table 10.5 contains the census data from 1790 to 1940.

TABLE 10.5 *US Population in millions, 1790–1940*

Year	Population	Year	Population	Year	Population
1790	3.9	1850	23.1	1910	92.0
1800	5.3	1860	31.4	1920	105.7
1810	7.2	1870	38.6	1930	122.8
1820	9.6	1880	50.2	1940	131.7
1830	12.9	1890	62.9		
1840	17.1	1900	76.0		

You might note that the population is given only to the nearest 0.1 million (or 100,000), alth the population is reported by the US Census Bureau down to the last digit (for example, 131,66! in 1940). We have rounded off because census figures are notoriously inaccurate. For examp the census of 1990, New York City claimed the census had missed a million people in that city ε Thus, giving more digits in the population does not necessarily give more accuracy.

Let us concentrate first on the relative growth rate, $(dP/dt)/P$, of the US population from to 1860. If we want to estimate $(dP/dt)/P$ in 1830, we take the rate of change in the popu and divide it by the population itself:[8]

$$\begin{aligned}\frac{1}{P}\frac{dP}{dt} &\approx \frac{1}{\text{Population in 1830}} \cdot \frac{\text{Population in 1840} - \text{Population in 1830}}{10 \text{ years}} \\ &= \frac{1}{12.9} \cdot \frac{17.1 - 12.9}{10} = 0.0326 = 3.26\%.\end{aligned}$$

Similar calculations for 1790, 1800, . . . , 1850 give the percentages in Table 10.6:

TABLE 10.6 *Rough estimates of yearly growth rate of US population*

Year	1790	1800	1810	1820	1830	1840	185
Relative growth rate	3.59%	3.58%	3.33%	3.44%	3.26%	3.57%	3.53

These percentages are pretty close. The relative growth rate, while not precisely const nearly so. In fact, political and economic events such as war or recession affect the populati we don't expect the growth rate to be exactly constant.

The simplest model for population growth is to assume that the relative growth rate *is* co in other words

$$\frac{1}{P}\frac{dP}{dt} = k,$$

[8] In Problem 12 at the end of this section we look at an alternative way of estimating the relative growth rate using data from 1820 as well as from 1840.

where k is the continuous growth rate. This is equivalent to assuming that the population grows exponentially:

$$P = P_0 e^{kt}.$$

What should we take for the value of k? One possibility would be the average of the percentages we just calculated, namely 3.47%. However, there is a serious objection to using this percentage as an estimate for k. Remember that k is a continuous growth rate, but the populations are given at 10-year intervals. A 10-year population growth of 34.7% doesn't come from a continuous yearly rate of 3.47%. (This would be ignoring the effects of compounding.) If the population increases by a factor of 34.7% in 10 years, then $P(10)/P_0 = 1.347$, where $P(10)$ is the population when $t = 10$. Assuming that $P = P_0 e^{kt}$, we need k to satisfy

$$\frac{P(10)}{P_0} = e^{k \cdot 10} = 1.347.$$

Now we can find the continuous growth rate, k:

$$k = \frac{\ln(1.347)}{10} = 0.0298.$$

Let's compare predicted and actual values if we model the US population by the differential equation

$$\frac{dP}{dt} = 0.0298P.$$

We start with initial population $P_0 = 3.9$ in 1790. Notice that this says we will consider 1790 as time $t = 0$, so 1800 is $t = 10$, and 1810 is $t = 20$, etc. The solution to the differential equation is

$$P = 3.9e^{0.0298t}.$$

If we put $t = 0, 10, 20, \ldots, 70$ into this function we get the populations predicted by our model for the years 1790, 1800, ..., 1860. Table 10.7 contains the comparison to the actual populations.

TABLE 10.7 *Predicted versus actual US population 1790–1860 (exponential model)*

Year	Actual	Predicted	Year	Actual	Predicted
1790	3.9	3.9	1830	12.9	12.8
1800	5.3	5.3	1840	17.1	17.3
1810	7.2	7.1	1850	23.1	23.3
1820	9.6	9.5	1860	31.4	31.4

The agreement is remarkable. Of course, since we used the data from the entire 70 year period to estimate k, we should expect good agreement throughout that period. What is surprising is that if we had used only the populations in 1790 and 1800 to estimate k, the predictions are still quite good. Let's find a new value of k using only the 1790 and 1800 data and compare predictions. The 10 year growth from 1790 to 1800 is 35.9% so $k = \ln(1.359)/10 = 0.0307$. We would then predict the population in 1860 to be

$$3.9e^{0.0307(70)} = 33.4,$$

which is within about 6% of the actual population of 31.4. It is remarkable that a person in 1800 could accurately predict what the population of the US would be 60 years later, especially considering all the wars, recessions, epidemics, additions of new territory, and immigration that took place from 1800 to 1860.

Predictions From The Exponential Model

The grim predictions of the exponential model are reflected in the ideas of Thomas Malthus, an early nineteenth-century clergyman and political philosopher, who believed that, if unchecked, a population would grow exponentially, whereas the food supply would grow linearly, and therefore the population would eventually outstrip the food supply.

Interestingly enough, an exponential model has fit the growth of world population and population of many regions remarkably well for decades, even centuries. However, the model m break down at some point because it predicts that the population will continue to grow without bou as time goes on—and this cannot be true forever. Eventually the effects of crowding, emigrati disease, war, and lack of food will have to curb growth. In searching for an improvement, then, should look for a model whose solution is approximately an exponential function for small val of the population, but which levels off later.

How to Estimate *dP/dt* from Data

If, as is often the case, all we know about a population, P, is its values at certain points in time, have to approximate dP/dt by $\Delta P/\Delta t$. However there are several different ways this approxima can be made. As in the example above, we can say

$$\frac{dP}{dt} \text{ at } 1830 \approx \frac{\text{Population in 1840} - \text{Population in 1830}}{10}.$$

However we could equally well have said

$$\frac{dP}{dt} \text{ at } 1830 \approx \frac{\text{Population in 1830} - \text{Population in 1820}}{10}.$$

Both of these are called *one-sided estimates* because they involve using the population to one of 1830 but not the other. The first one involving 1840 is the forward, or right-hand, estimate: second one involving 1820 is the backward, or left-hand estimate. In general, both are equally g or bad estimates for dP/dt. A more accurate approximation can be obtained by averaging the one-sided estimates, giving

$$\frac{dP}{dt} \text{ at } 1830 \approx \frac{1}{2}\left(\frac{\text{Population in 1840} - \text{Population in 1830}}{10} + \frac{\text{Population in 1830} - \text{Population in 182}}{10}\right.$$

When we add the two fractions the population in 1830 cancels and we get the two-sided estir so called because it involves data on both sides of 1830:

$$\frac{dP}{dt} \text{ at } 1830 \approx \frac{1}{2}\left(\frac{\text{Population 1840} - \text{Population 1820}}{10}\right) = \left(\frac{\text{Population 1840} - \text{Population 1820}}{20}\right.$$

In the exponential model above we used a one-sided estimate. This turned out to give results after adjusting from 10-year to continuous growth rates. If we had been unable to get acc predictions using one-sided estimates, we might have tried two-sided estimates instead. In the lo model below, we use two-sided estimates, as they turn out to give noticeably better prediction

The US Population: 1790--1940

The exponential model for the US population works well for reasonable periods of time exponential growth cannot go on forever. For example, if we tried to predict the population o US in 1990 using the exponential model we developed above with $k = 0.0298$, we would get

$$\text{US Population in 1990} = 3.9e^{(0.0298)(200)} = 1{,}512 \text{ million},$$

which is far from the actual figure of around 250 million.

The problem is that growth rate in the US population between 1790 and 1860 did not stay co in later decades. The 10-year percentage growths[9] from 1860 to 1930 are listed in Table 10.8:

TABLE 10.8 *Estimated 10-year growth rate of US population, 1860–1930*

Year	1860	1870	1880	1890	1900	1910	1920	1
Relative growth rate	22.9%	30.1%	25.3%	20.8%	21.1%	14.9%	16.2%	7

[9]Calculated using one-sided forward estimates, so that $\frac{1}{P}\frac{dP}{dt}$ at 1860 $\approx \frac{\text{Population in 1870} - \text{Population in 1860}}{10(\text{Population in 1860})}$.

These figures are nothing like those during the period 1790 to 1860, where they hovered around 34%. The dramatic drop to 22.9% for the decade 1860–1870 is explainable by the Civil War (but don't ascribe the entire drop to deaths of the war—see Problem 15, page 541). The growth rate goes back up during the decade 1870–1880, but by 1890–1900 it has dropped below the rate during the Civil War. There is a slight increase in the rate during the immigrations of 1900 to 1910, another drop during World War I, a small bounce back, and finally the rate plummets during the recession of the 1930s. Notice that the effect of the recession is more dramatic than the effect of wars, which suggests that it is a decrease in birth rate rather than an increase in death rate that is a major factor in declines in the relative population growth rate.

Our exponential model gives accurate predictions up to 1860. But for the years following 1860, the exponential model is inadequate. We look for a new model that will take into account the effects of overcrowding. Because of the effects of crowding, we expect the relative growth rate to decrease as the population increases. Thus we look at how $(dP/dt)/P$ changes as P changes. This time we use a two-sided estimate for dP/dt. For example:

$$\frac{dP}{dt} \text{ at } 1830 \approx \frac{\text{Population in 1840} - \text{Population 1820}}{20}.$$

Table 10.9 contains estimates for $(dP/dt)/P$ computed this way for some years between 1790 and 1940. Comparing the last two columns suggests that the values in the last column may be an approximately linear function of P. To get a better picture of how $(dP/dt)/P$ varies with P, we plot $(dP/dt)/P$ versus P and see whether the points lie on a line. Figure 10.45 shows the scatterplot of the points together with the line that best fits the points. The equation for the line, which fits quite well, is

$$\frac{1}{P}\frac{dP}{dt} = 0.0318 - 0.000170P.$$

TABLE 10.9 *Some estimates for* $\frac{1}{P}\frac{dP}{dt}$.

Year	P	$(P(t+10) - P(t-10))/(20P)$
1860	31.4	0.0245
1890	62.9	0.0205
1910	92.0	0.0161
1930	122.8	0.0106

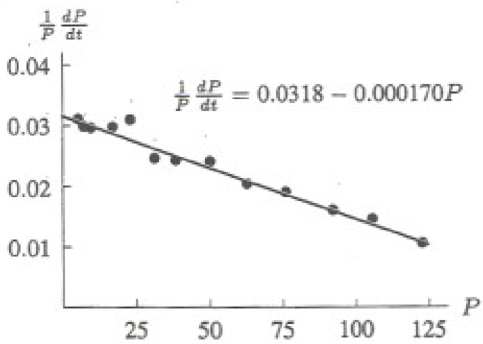

Figure 10.45: Scatterplot for US census data of $\frac{1}{P}\frac{dP}{dt}$ versus P.

Therefore, in our new model P satisfies the differential equation

$$\frac{dP}{dt} = 0.0318P - 0.000170P^2.$$

This is known as a *logistic equation*. Its slope field is shown in Figure 10.46, together with the solution with $P(0) = 3.9$ superimposed. (Notice that $t = 0$ in 1790.)

The most striking difference in this model compared with the exponential model is that it predicts that the US population will level off somewhere below 200 million. The population will continue to grow until $dP/dt = 0$, which occurs when

$$0 = 0.0318P - 0.000170P^2$$

so

$$P = 0 \quad \text{or} \quad P \approx 187 \text{ million.}$$

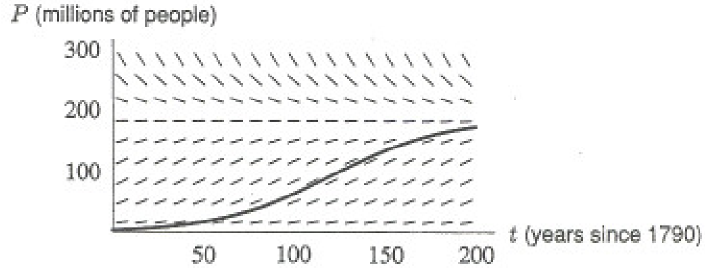

Figure 10.46: Solution to $\frac{dP}{dt} = 0.0318P - 0.000170P^2$ with $P(0) = 3.9$

Looking at the shape of the solution curve in Figure 10.46, we see that initially the population grov faster and faster and then slows down as the limiting value of 187 is approached; the fastest grow rate appears to be about half-way to the limiting value.

Later on in this section we derive the formula for the solution to the logistic equation. For no' you can check by substitution that the function

$$P = \frac{187}{1 + 47e^{-0.0318t}}$$

is a solution to our logistic equation modeling the US population. (The numbers 187 and the 47 a not exact values, but have been rounded.) The values predicted by this equation for P agree ve well with the actual populations up to 1940. See Table 10.10. During the period from 1700 to 194 the largest deviation is about 3% in 1840 and 1870 (the Civil War accounts for the second one). A other errors are less than 2%.

Of course, the final test is how well our model, based on data from 1790 to 1940, predicts tl population in the "future," 1950 to 1990. Table 10.10 contains the predicted and actual data.

TABLE 10.10 *Predicted versus actual US population in millions, 1790–1980 (logistic model)*

Year	Actual	Predicted	Year	Actual	Predicted	Year	Actual	Predicted
1790	3.9	3.9	1860	31.4	30.8	1930	122.8	120.8
1800	5.3	5.3	1870	38.6	39.9	1940	131.7	133.7
1810	7.2	7.2	1880	50.2	50.7	1950	150.7	145.0
1820	9.6	9.8	1890	62.9	63.3	1960	179.3	154.4
1830	12.9	13.2	1900	76.0	77.2	1970	203.3	162.1
1840	17.1	17.7	1910	92.0	91.9	1980	226.5	168.2
1850	23.1	23.5	1920	105.7	106.7	1990	248.7	172.9

The fit between predicted and actual population values is clearly not good from 1950 on. Desp World War II, which undoubtedly depressed population growth between 1942 and 1945, in the l half of the 1940s the US population surged, wiping out in five years a deficit caused by 15 years depression and war. The 1950s saw a population growth of 28 million, leaving our logistic moc in the dust. This surge in population is referred to as the baby boom. All one can say is that bas on 150 years of data, what happened in the US in the 20 years after World War II was complet without precedent. The baby boom could well end up being one of the most important sociologi events of the twentieth century in the United States, and its consequences will be felt for many ye to come.

Once again we have reached a point where our model is no longer useful. This should I lead you to believe that a reasonable mathematical model cannot be found; rather it should se to point out that no model is perfect and that when one model fails, we seek a better one. Just we abandoned the exponential model in favor of the logistic model for the US population, we co look further. (See Problems 9 and 10 on page 540.)

he Logistic Model

The logistic model we used to model the US population from 1790 to 1940 assumed that the relative growth rate of the population was a linearly decreasing function of P:

$$\frac{1}{P}\frac{dP}{dt} = k - aP.$$

(We took $k = 0.0318$ and $a = 0.000170$). For small P, we have approximately

$$\frac{1}{P}\frac{dP}{dt} \approx k.$$

The solution to this equation is an exponential function. This is why an exponential model fit the US population well during the years 1790-1860 when the population was relatively small. In the logistic model, as P increases, the relative growth rate decreases to zero; it reaches zero when P is given by

$$k - aP = 0.$$

Solving for P, we get

$$P = \frac{k}{a}.$$

This is the limiting value of the population, which we call L:

$$L = \frac{k}{a}.$$

The value L is called the *carrying capacity* of the environment, and represents the largest population the environment can support. Writing $a = k/L$, the logistic equation becomes

$$\frac{1}{P}\frac{dP}{dt} = k - \frac{k}{L}P$$

$$\frac{dP}{dt} = kP - \frac{k}{L}P^2$$

or

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

This is the general logistic differential equation, first proposed as a model for population growth by the Belgian mathematician P. F. Verhulst in the 1830s.

Qualitative Solution to the Logistic Equation

Figure 10.47 shows the slope field and characteristic *sigmoid*, or S-shaped, solution curve for the logistic model. Notice that for each fixed value of P, that is, along each horizontal line, the slopes are all the same because dP/dt depends only on P and not on t. The slopes are small near $P = 0$ and near $P = L$; they are steepest around $P = L/2$. For $P > L$, the slopes are negative, meaning that if the population is above the carrying capacity, the population will decrease.

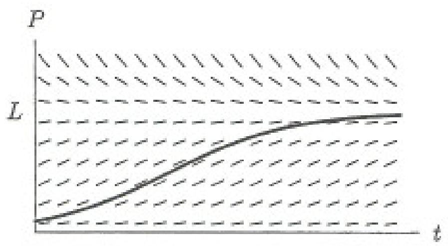

Figure 10.47: Slope field for $\frac{dP}{dt} = kP(1 - \frac{P}{L})$

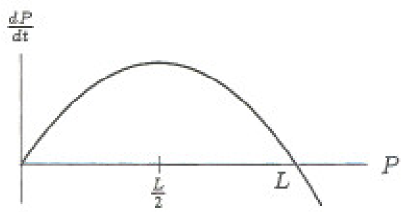

Figure 10.48: $\frac{dP}{dt} = kP(1 - \frac{P}{L})$

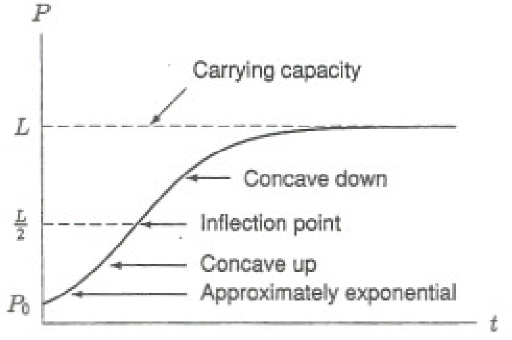

Figure 10.49: Logistic growth with inflection point

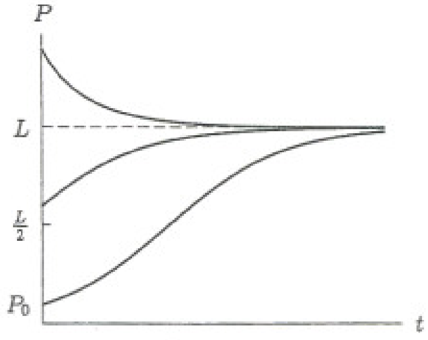

Figure 10.50: Solutions to the logistic equation

We can locate precisely the inflection point where the slopes are greatest using the graph dP/dt against P in Figure 10.48. The graph is a parabola because dP/dt is a quadratic functi of P. The horizontal intercepts are at $P = 0$ and $P = L$, so the maximum, where the slope greatest, is at $P = L/2$. The graph in Figure 10.48 also tells us that for $0 < P < L/2$, the slo dP/dt is positive and increasing, so the graph of P against t is concave up. (See Figure 10.49.) F $L/2 < P < L$, the slope dP/dt is positive and decreasing, so the graph of P against t is conca down. For $P > L$, the slope dP/dt is negative, so the graph of P against t is decreasing.

If $P = 0$ or $P = L$, there is an equilibrium solution (not a very interesting one if $P =$ Figure 10.50 shows that $P = 0$ is an unstable equilibrium because solutions which start near 0 mc away. However, $P = L$ is a stable equilibrium.

The Analytic Solution to the Logistic Equation

We have already obtained a lot of information about logistic growth without finding a formula the solution. However, the equation can be solved analytically by separating variables:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) = kP\left(\frac{L-P}{L}\right)$$

giving

$$\int \frac{dP}{P(L-P)} = \int \frac{k}{L}\, dt.$$

We can integrate the left side using the integral tables (Formula 26), or by rewriting

$$\frac{1}{P(L-P)} = \frac{1}{L}\left(\frac{1}{P} + \frac{1}{L-P}\right).$$

Thus, we have

$$\int \frac{1}{L}\left(\frac{1}{P} + \frac{1}{L-P}\right) dP = \int \frac{k}{L}\, dt.$$

Canceling the constant L, we get

$$\int \left(\frac{1}{P} + \frac{1}{L-P}\right) dP = \int k\, dt$$

which can be integrated to give

$$\ln|P| - \ln|L-P| = kt + C.$$

Multiplying through by (-1) and using the fact that $\ln M - \ln N = \ln(M/N)$, we have

$$\ln\left|\frac{L-P}{P}\right| = -kt - C.$$

Exponentiating both sides gives

$$\left|\frac{L-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt},$$

so

$$\frac{L-P}{P} = Ae^{-kt} \quad \text{where} \quad A = \pm e^{-C}.$$

We find A by substituting $P = P_0$ when $t = 0$, which gives

$$\frac{L-P_0}{P_0} = Ae^0 = A.$$

Thus

$$\frac{L-P}{P} = Ae^{-kt} \quad \text{where} \quad A = \frac{L-P_0}{P_0}.$$

Since $(L-P)/P = (L/P) - 1$, we have

$$\frac{L}{P} = 1 + Ae^{-kt}$$

giving the formula for the logistic curve:

$$P = \frac{L}{1+Ae^{-kt}} \qquad \text{where} \quad A = \frac{L-P_0}{P_0}.$$

Problems for Section 10.7

1. Assuming that Switzerland's population is growing exponentially at a continuous rate of 0.2% a year and that its 1988 population was 6.6 million, write an expression for the population as a function of time in years. (Let $t = 0$ in 1988.)
2. Consider the logistic model

$$\frac{dP}{dt} = 3P - 3P^2.$$

(a) On the slope field in Figure 10.51, sketch three solution curves showing different types of behavior.
(b) Is there a stable value of the population? If so, what is it?
(c) Describe the meaning of the shape of the solution curves for the population: Where is P increasing? Decreasing? What happens in the long run? Are there any inflection points? Where? What do they mean for the population?

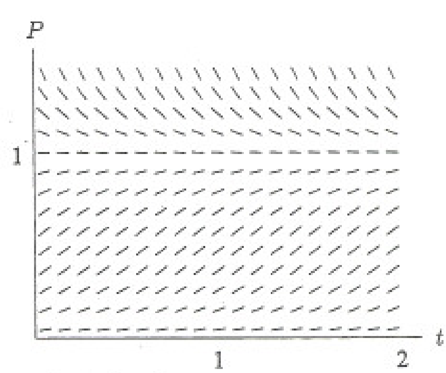

Figure 10.51: Slope field for $dP/dt = 3P - 3P^2$.

(d) Sketch a graph of dP/dt against P. Where is dP/dt positive? Negative? Zero? Maximum? How do your observations about dP/dt explain the shapes of your solution curves?

3. The total number of people infected with a virus often grows like a logistic curve. Suppose that 10 peopl originally have the virus, and that in the early stages of the virus (with time, t, measured in weeks), th number of people infected is increasing exponentially with $k = 1.78$. It is estimated that, in the lon run, approximately 5000 people become infected.
   (a) Use this information to find a logistic function to model this situation.
   (b) Sketch a graph of your answer to part (a).
   (c) Use your graph to estimate the length of time until the rate at which people are becoming infecte starts to decrease. What is the vertical coordinate at this point?

4. Table 10.11 gives the percentage, P, of households with a VCR, as a function of year.
   (a) Explain why a logistic model is a reasonable one to use for this data.
   (b) Use the data to estimate the point of inflection of P. What limiting value L does this point c inflection predict? Does this limiting value appear to be accurate given the percentages for 199 and 1991?
   (c) The best logistic equation for this data turns out to be the following. What limiting value does thi model predict?

$$P = \frac{75}{1 + 316.75e^{-0.699t}}.$$

TABLE 10.11 *Percentage of households with a VCR*

Year	1978	1979	1980	1981	1982	1983	1984
$P(\%)$	0.3	0.5	1.1	1.8	3.1	5.5	10.6
Year	1985	1986	1987	1988	1989	1990	1991
$P(\%)$	20.8	36.0	48.7	58.0	64.6	71.9	71.9

5. The growth of a certain animal population is governed by the equation

$$\frac{1000}{P}\frac{dP}{dt} = 100 - P,$$

where $P(t)$ is the number of individuals in the colony at time t. The initial population is known to b 200 individuals. Sketch a graph of $P(t)$. Will there ever be more than 200 individuals in the colon Will there ever be fewer than 100 individuals? Explain.

6. It is of considerable interest to policy makers to model the spread of information through a populatio For example, various agricultural ministries use models to help them understand the spread of technic innovations or new seed types through their countries. Two models, based on how the information spread, are given below. Assume the population is of a constant size M.
   (a) If the information is spread by mass media (TV, radio, newspapers), the rate at which informati is spread is believed to be proportional to the number of people not having the information at th time. Write a differential equation for the number of people having the information by time Sketch a solution assuming that no one (except the mass media) has the information initially.
   (b) If the information is spread by word of mouth, the rate of spread of information is believed be proportional to the product of the number of people who know and the number who don Write a differential equation for the number of people having the information by time t. Sket the solution for the cases in which
      (i) no one
      (ii) 5% of the population
      (iii) 75% of the population

   knows initially. In each case, when is the information spreading fastest?

7. The population of a species of elk on Reading Island in Canada has been monitored for some yea When the population was 600, the relative birth rate was found to be 35% and the relative death r was 15%. As the population grew to 800, the corresponding figures were 30% and 20%. The island isolated so there is no hunting or migration.
   (a) Write a differential equation to model the population as a function of time. Assume that relati growth rate is a linear function of population.

(b) Find the equilibrium size of the population. Today there are 900 elk on Reading Island. How do you expect the population to change in the future?

(c) Oil has been discovered on a neighboring island and the oil companies want to move 450 elk of the same species to Reading Island. What effect would this move have on the elk population on Reading Island in the future?

(d) Assuming the elk are moved to Reading Island, sketch the population on Reading Island as a function of time. Start before the elk are transferred and continue for some time into the future. Comment on the significance of your results.

8. Many organ pipes in old European churches are made of tin. In cold climates such pipes can be affected with *tin pest*, when the tin becomes brittle and crumbles into a grey powder. This transformation can appear to take place very suddenly because the presence of the grey powder encourages the reaction to proceed. At the start, when there is little grey powder, the reaction proceeds slowly. Similarly, toward the end, when there is little metallic tin left, the reaction is also slow. In between, however, when there is plenty of both metallic tin and powder, the reaction can be alarmingly fast.

   Suppose that the rate of the reaction is proportional to the product of the amount of tin left and the quantity of grey powder, p, present at time t. Assume also that when metallic tin is converted to grey powder, its mass does not change.

   (a) Write a differential equation for p. Let the total quantity of metallic tin present originally be B.

   (b) Sketch a graph of the solution $p = f(t)$ if there is a small quantity of powder initially. How much metallic tin has crumbled when it is crumbling fastest?

   (c) Suppose there is no grey powder initially. (For example, suppose the tin is completely new.) What does this model predict will happen? How do you reconcile this with the fact that many organ pipes do get tin pest?

9. (a) In the text we fitted a logistic model to the US population from 1790–1940. In this problem, we try to fit a logistic equation to the US population all the way from 1790 to 1990. No logistic equation fits the data exactly over this entire period, but we can use the method of page 534 to find an equation that does reasonably well throughout.

   To fit a logistic equation to the data in Table 10.12, we estimate the relative growth rate, $(dP/dt)/P$, and plot it against P. To approximate $(dP/dt)/P$, calculate $(\Delta P/\Delta t)/P$ from the data for seven fairly spread-out points. Draw a reasonable line (by eye) through your points, and thus estimate k and a for the equation $(dP/dt)/P = k - aP$.

TABLE 10.12 *US population in millions, 1790–1990*

Year	Population	Year	Population	Year	Population
1790	3.9	1860	31.4	1930	122.8
1800	5.3	1870	38.6	1940	131.7
1810	7.2	1880	50.2	1950	150.7
1820	9.6	1890	62.9	1960	179.0
1830	12.9	1900	76.0	1970	205.0
1840	17.1	1910	92.0	1980	226.5
1850	23.1	1920	105.7	1990	248.7

   (b) What does this model predict about the US population in the long run?

10. (a) In Problem 9 we saw that a logistic model cannot be made to fit the US population very closely, because the points on the graph of $(dP/dt)/P$ against P are not exactly on a line. In this problem we try another model. This time we assume that $(dP/dt)/P$ is a linear function of t, so you plot $(dP/dt)/P$ against t (using the same approximate values you calculated in Problem 9). Put a line through the points by eye, and estimate a and b to fit the equation

$$\frac{1}{P}\frac{dP}{dt} = a - bt.$$

   (b) When, if ever, does this model predict that the US population will be at its maximum?

   (c) Solve the differential equation and sketch its solution.

11. An alternative method of finding the analytic solution to the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

uses the substitution $P = 1/u$.

(a) Show that

$$\frac{dP}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

(b) Rewrite the logistic equation in terms of u and t, and solve for u in terms of t.

(c) Using your answer to part (b), find P as a function of t.

12. On page 531, we used one-sided estimates for dP/dt to fit an exponential model to the US popula from 1790–1860. In this problem we use two-sided estimates for the years 1800–1850. Suppose the US population in millions.

(a) Estimate dP/dt by the symmetric difference quotient $\left(P(t+10) - P(t-10)\right)/20$. Then c pute $(dP/dt)/P$ for each of these years and average them to estimate k for the exponential m $dP/dt = kP$. Compare your value of k with the estimate $k \approx 3.47\%$ obtained on page 53 using $\left(P(t+10) - P(t)\right)/10$ for dP/dt.

(b) Using your answer to part (a), compute k as the continuous rate of change that leads to the obse 10-year percentage change. Then compare your k with the value $k \approx 2.98\%$ obtained on page

13. This section suggested two ways of estimating dP/dt: the one-sided $\left(P(t+h) - P(t)\right)/h$, the two-sided $\left(P(t+h) - P(t-h)\right)/(2h)$. Let $f(x) = x^3$. Consider the approximations $f'(2$ $\left(f(2+h) - f(2)\right)/h$ and $f'(2) \approx \left(f(2+h) - f(2-h)\right)/2h$ for $h = 0.1, 0.01, 0.001$. Whi the better approximation? Is there a pattern to the errors in the approximation as you decrease h? I describe it.

14. Show that if $f(x) = x^2$, then $(f(x+h) - f(x-h))/(2h) = f'(x)$ for any value of h. This shows if $f(x) = x^2$, the two-sided estimate for the derivative of $f(x)$ is exactly equal to the derivative.

15. Estimate the US population in 1870 from the 1860 population of 31.4 million, assuming tha population increased at the same percentage rate during the 1860s as it did in the decades pre to 1860 (about 34.7% each decade). Compare your estimate with the actual 1870 population of million. Find some estimate of the number of people who died in the Civil War. Does the numb deaths in the Civil War explain the shortfall in the actual population in 1870? What else might influ the shortfall?

16. Another way to estimate the limiting population L for the logistic differential equation is to note the derivative dP/dt is largest when $P = L/2$. Compute dP/dt for the US census data for 1790– using the two-sided difference quotient $\left(P(t+10) - P(t-10)\right)/20$. Estimate L by doublin population when dP/dt is largest. Compare this estimate with the estimate $L = 187$ given in the te

Any population, P, for which we can ignore immigration, satisfies

$$\frac{dP}{dt} = \text{Birth rate} - \text{Death rate}.$$

For organisms which need a partner for reproduction but rely on a chance encounter for meeting a mat birth rate is proportional to the square of the population. Thus, the population of such a type of org satisfies a differential equation of the form

$$\frac{dP}{dt} = aP^2 - bP \quad \text{with } a, b > 0.$$

Problems 17–19 investigate the solutions to such an equation.

17. Consider the equation

$$\frac{dP}{dt} = 0.02P^2 - 0.08P.$$

(a) Sketch the slope field for this differential equation for $0 \le t \le 50$, $0 \le P \le 8$.

(b) Use your slope field to sketch the general shape of the solutions to the differential eq satisfying the following initial conditions:

(i) $P(0) = 1$ (ii) $P(0) = 3$ (iii) $P(0) = 4$ (iv) $P(0) = 5$

(c) Are there any equilibrium values of the population? If so, are they stable?

18. Consider the equation

$$\frac{dP}{dt} = P^2 - 6P.$$

(a) Sketch a graph of dP/dt against P for positive P.
(b) Use the graph you drew in part (a) to sketch the approximate shape of the solution curve with $P(0) = 5$. To do this, consider the following question. For $0 < P < 6$, is dP/dt positive or negative? What does this tell you about the graph of P against t? As you move along the solution curve with $P(0) = 5$, how does the value of dP/dt change? What does this tell you about the concavity of the graph of P against t?
(c) Use the graph you drew in part (a) to sketch the solution curve with $P(0) = 8$.
(d) Describe the qualitative differences in the behavior of populations with initial value less than 6 and initial value more than 6. Why do you think $P = 6$ is called the *threshold population*?

19. Consider a population satisfying

$$\frac{dP}{dt} = aP^2 - bP \quad \text{with constants } a,\ b > 0.$$

(a) Sketch a graph of dP/dt against P.
(b) Use this graph to sketch the shape of solution curves with various initial values. Use your graph from part (a) to decide where dP/dt is positive or negative, and where it is increasing or decreasing. What does this tell you about the graph of P against t?
(c) Why is $P = b/a$ called the threshold population? What happens if $P(0) = b/a$? What happens in the long-run if $P(0) > b/a$? What if $P(0) < b/a$?

10.8 SECOND-ORDER DIFFERENTIAL EQUATIONS: OSCILLATIONS

A Second-Order Differential Equation

When a body moves freely under gravity, we know that

$$\frac{d^2s}{dt^2} = -g,$$

where s is the height of the body above ground at time t and g is the acceleration due to gravity. To solve this equation, we first integrate to get the velocity, $v = ds/dt$:

$$\frac{ds}{dt} = -gt + v_0,$$

where v_0 is the initial velocity. Then we integrate again, giving

$$s = -\frac{1}{2}gt^2 + v_0t + s_0,$$

where s_0 is the initial height.

The differential equation $d^2s/dt^2 = -g$ is called *second order* because the equation contains a second derivative but no higher derivatives. The general solution to a second-order differential equation will be a family of functions with two parameters, here v_0 and s_0. Finding values for the two constants corresponds to picking a particular function out of this family.