Test 2: Math 114

Calculus 1 (Experienced) Tuesday, October 25, 2005

Name: KEY

Prof. Buckmire (10:30am)
Prof. Gallegos (1:30pm)
(circle one)

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0. **NO CALCULATORS** are to be used on this exam. Please refer to the sheet of helpful values on the last page.
1. Read each question carefully and be sure to answer appropriately and thoroughly. When it makes sense, **answer in complete sentences.**
2. Partial credit will be given, but only if we can see the correct parts. So **show all of your work** and **explain your answers!**
3. Although the exam is designed to be completed in 55 minutes you may have from 7:00 to 9:00pm.
4. Before you hand in your exam please sign the pledge below.
5. Make sure your "Blue Notes" is attached.
6. Relax and enjoy... (Good Luck!)

Pledge: I, ____________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		1
2		19
3		30
4		20
5		30
BONUS		5
Total		**100**

1. The Joke Problem. *(1 point.)* Write down your favorite math joke–or ar
(that we could tell in class).

Q: Why did the ~~chicken~~ cow
cross the road?

A: To get to the udder side.

2. The Related Rates Problem. *(19 points.)* Brain size $b(x)$, in grams, and IQ $I(x)$, measured in IQ "points", in chimpanzees are a function of their age x in years. They are related by the equation

$$12\ln^2(b) - I^{2/3} = \text{constant}.$$

In a particular chimpanzee, at age 5 whose brain size is 250 grams, the ratio of brain size to IQ is 2 and the brain size is decreasing at 3 grams/year. How fast, in IQ points, is the chimpanzee's IQ decreasing? **Recall that there is a page of values on the last page of this test which you may find helpful.**

$\frac{dI}{dx}$ = rate of change of IQ in points/year

$\frac{db}{dx}$ = rate of change of brain size in g/year

$$\frac{d}{dx}\left[12\ln^2(b) - I^{2/3}\right] = \frac{d}{dx}(\text{constant})$$

$$12\frac{d}{dx}(\ln b(x))^2 - \frac{d}{dx}\left[(I(x))^{2/3}\right] = 0$$

Chain Rule: $12\cdot 2\cdot \frac{\ln b(x)}{b(x)}\cdot\frac{db}{dx} - \frac{2}{3}I(x)^{-1/3}\frac{dI}{dx} = 0$

$$24\frac{\ln b}{b}\frac{db}{dx} = \frac{2}{3}I^{-1/3}\frac{dI}{dx}$$

$$\frac{3}{2}\cdot 24\frac{\ln b}{b}\frac{dB}{dx}\cdot\frac{1}{I^{-1/3}} = \frac{dI}{dx}$$

$$36\frac{\ln b}{\cancel{\ln} b}\frac{db}{dx}\cdot I^{1/3} = \frac{dI}{dx}$$

When $x=5$, $\frac{db}{dx} = -3$ g/year $\quad \frac{b(5)}{I(5)} = 2$, $b(5) = 250$g

$I(5) = \frac{b(5)}{2} = \frac{250\text{g}}{2} = 125$g

$$\frac{dI}{dx} = 36\cdot\frac{\ln(250)}{250}(-3)\cdot(125)^{1/3}$$

$$= -108\cdot\ln(250)\cdot 5/250$$

$$= -\frac{540\ln(250)}{250}\ \cancel{\text{g/year}}\ \frac{\text{points}}{\text{year}} \approx -11 \text{ points/year}$$

3. The Limit Problem. Suppose a mystery function $\Omega(x)$ has the following properties:

PROPERTY 1: $\Omega(a+b) = \Omega(a)\Omega(b)$

PROPERTY 2: $\lim_{b\to 0} \frac{\Omega(b)-1}{b} = 1.$

(a) *(20 points)* Use the limit definition of the derivative to find a formula for $\Omega'(a)$ in terms of $\Omega(a)$. Clearly show each step. For full credit, give a reason for each step.

$$\Omega'(a) = \lim_{h\to 0} \frac{\Omega(a+h)-\Omega(a)}{h}$$ (def^n of derivative)

$$= \lim_{h\to 0} \frac{\Omega(a)\cdot\Omega(h)-\Omega(a)}{h}$$ (Property 1)

$$= \lim_{h\to 0} \frac{\Omega(a)\cdot(\Omega(h)-1)}{h}$$ (factoring)

$$= \lim_{h\to 0} \Omega(a) \cdot \lim_{h\to 0} \frac{\Omega(h)-1}{h}$$ (limit of product is product of limits)

$$= \Omega(a)\cdot 1$$ (property 2) (limit of constant = constant)

$$\Omega'(a) = \Omega(a)$$

(b) *(10 points)* The function Ω is actually an elementary function you know well. What is it? Give reasons supporting your choice.

The function whose derivative is itself is the exponential function $e^x = \Omega(x)$.

$$\Omega(a+b) = e^{a+b} = e^a \cdot e^b = \Omega(a)\,\Omega(b)$$ satisfies Property 1

$$\lim_{b\to 0}\frac{e^b-1}{b} = \lim_{b\to 0}\frac{e^b}{1} = 1$$ satisfies Property 2

L'Hospital's Rule

4. The Verbal Problem. *(20 points.)* Three students, Madison, Sydney and Harper have been surfing the internet and have discovered some interesting properties about the absolute value function, $f(x) = |x|$.

Harper: So this website says that the $|x|$ function is continuous everywhere. I don't believe it–look at what happens to the graph of the function at $x = 0$, no way it's continuous there too!

Madison: I believe everything I read on the web. In fact, I think I read somewhere else that $|x|$ is also differentiable everywhere, so that must be true too.

Sydney: Hello, do you not remember the lab we had on this topic? The absolute value function is continuous everywhere but not differentiable anywhere.

Harper: Well, I agree the absolute value function is not differentiable at the origin $(0, 0)$ since it's obviously not continuous there.

Sydney: But the absolute value function *is* continuous at the origin! I know it is because I can clearly sketch the graph near the origin without picking up my pencil. The reason why $|x|$ is not differentiable at $x = 0$ is because it is not locally linear there.

Madison: Well, to me, every point except the origin of the graph function looks exactly like a line as you zoom in. So I would say that it is locally linear everywhere except the origin, but local linearity is about having a tangent line–it has nothing to do with differentiability.

Write a short essay explaining at least five incorrect statements made by the students about the concepts of local linearity, differentiability and continuity, specifically as it applies to the absolute value function $|x|$. A good approach would be to define these concepts, describe how they apply in the case of the absolute value function and then say whether and why each of the statements made by the students is true or false. **You must be careful not to make any incorrect statements yourself in your explanations. PROOFREAD YOUR ANSWER FOR GRAMMATICAL AND FACTUAL ERRORS.**

Harper's 1st statement is incorrect: $|x|$ is continuous at $x=0$ (and everywhere else). $\lim_{x\to 0^-} |x| = \lim_{x \to 0^+} = |0| = 0$ proves $|x|$ is continuous at $x=0$.

$|x|$ is not differentiable everywhere, it is not differentiable at $x=0$, because the graph is not locally linear there and ~~the~~ different difference quotients will not produce the same answer at $x=0$, as seen in Lab 3.

Sydney overstates the case. $|x|$ is continuous everywhere ~~Bo~~ and differentiable everywhere EXCEPT $x=0$.

Harper still refuses to believe $|x|$ is continuous at $x=0$, but is correct in saying if $|x|$ is NOT CONTINUOUS THEN it's NOT DIFFERENTIABLE at $x=0$.

Sydney's second statement is completely correct.

Madison is correct about the meaning of local linearity, but tangent lines have everything to do with differentiability, the slope of the tangent lines IS the value of the derivative of the function at that point.

5. **The Computational Problem.** For each of the following problems, find the derivative of the function. State what rules you use and clearly show what steps you use to obtain the derivative function in each case. **DO NOT SIMPLIFY.**

(a) *(10 points.)* $f(t) = \frac{3t}{\sqrt{t}} + \frac{5}{t^2} + \ln(e) + t^{0.7}e^{4t^2}$; $f'(t) =$

$$f(t) = 3\sqrt{t} + 5t^{-2} + 1 + t^{0.7}e^{4t^2}$$

$$f'(t) = \underbrace{3\frac{1}{2\sqrt{t}} - 10t^{-3}}_{\text{POWER RULE}} + \underset{\uparrow}{0} + \underbrace{0.7t^{-0.3}e^{4t^2} + t^{0.7}e^{4t^2}\cdot 8t}_{\text{PRODUCT RULE}}$$

CONSTANT

(b) *(10 points.)* $g(x) = \sin\left(\frac{x+1}{x-1}\right) + \cos\left(\frac{3\pi}{2}\right)$; $g'(x) = \cos\left(\frac{x+1}{x-1}\right)\cdot\left(\frac{x+1}{x-1}\right)'$

$$= \cos\left(\frac{x+1}{x-1}\right)\cdot\left[\frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2}\right]$$

QUOTIENT RULE

(c) *(10 points.)* $h(u) = u\ln\left(\sin\left(\cos(u^{1/3})\right)\right)$; $h'(u) =$

$$h'(u) = 1\cdot\ln(\sin(\cos(u^{1/3}))) + u\cdot\frac{1}{\sin(\cos(u^{1/3}))}\cdot\cos(\cos(u^{1/3}))\cdot -\sin(u^{1/3})\cdot\frac{1}{3}u^{-2/3}$$

PRODUCT RULE

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CHAIN RULE (4 TIMES)

BONUS: *(5 points.)* In the graph below, the function $f(x) = e^x - 3$ is plotted along with the line $y = x$. Sketch the function $g(x) = f^{-1}(x)$ on the graph. Also, find a formula for $g(x)$ and evaluate its derivative when $x = 0$. **NOTE: Bonus Points are hard to get so be particularly careful to fully explain your answers and show your work and reasoning.**

$f(x) = e^x - 3 \quad f'(x) = e^x$

$y = e^x - 3$

$y + 3 = e^x$

$\ln(y+3) = \ln(e^x) = x$

$\ln(y+3) = x$

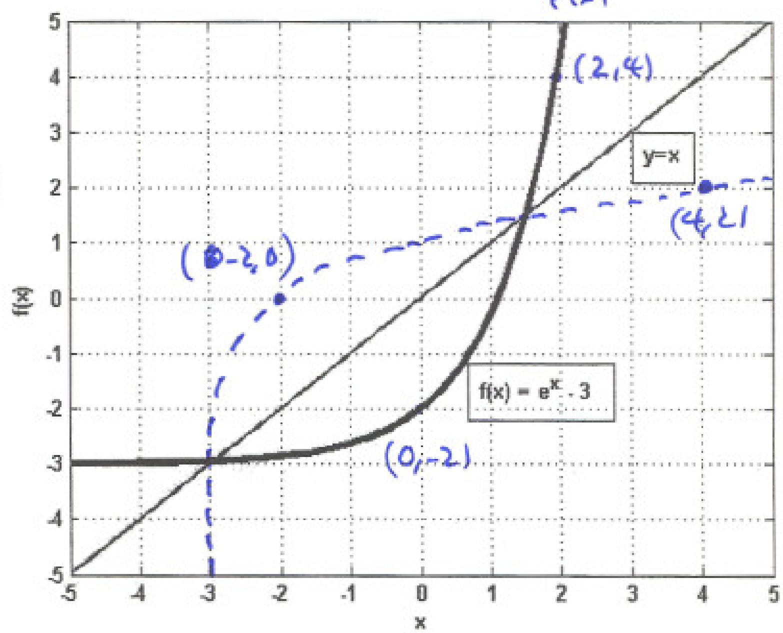
$\ln(x+3) = y$

(switch at end)

$f^{-1}(x) = \ln(x+3)$

$g(x) = \ln(x+3)$

$g(0) = \ln 3$



$$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{f'(\ln 3)} = \frac{1}{e^{\ln 3}} = \boxed{\frac{1}{3}}$$

Some Useful and Not-so-useful Values!

1. $\sqrt[3]{250} \approx 6.3$
2. $\sqrt[3]{125} = 5$
3. $\sqrt[3]{500} \approx 7.94$
4. $\sqrt[3]{1000} = 10$
5. $\sqrt[3]{8} = 2$
6. $\ln(250) \approx 5.52$
7. $\ln(125) \approx 4.83$
8. $\ln(500) \approx 6.21$
9. $\ln(1000) \approx 6.91$
10. $\ln(8) \approx 2.08$
11. $\cos(0) = 1$
12. $\sin(0) = 0$
13. $\tan(0) = 0$
14. $e^8 \approx 2980.96$
15. $e^{10} \approx 22026.47$