1. Holidays! *(1 point)* Draw or name your favorite Thanksgiving food item. (Alternately you could name your favorite doughnut–it depends which you like better!)

Turkey!!

2. Fun with Rate Equations. *(30 points.)*

a. For the rate equation

$$\frac{dx}{dt} = -5 + x^2,$$

find all equilibrium values, determine their stability and sketch representative solutions $x(t)$ for $t > 0$.

Equilibrium values occur at $\frac{dx}{dt} = 0$

$-5 + x^2 = 0$

$x^2 = 5$

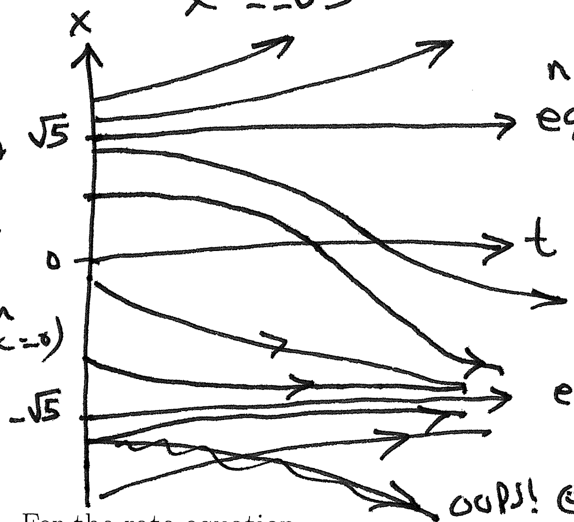
$x = \pm\sqrt{5}$

eq vals 1
x inc/dec
2 f''
1 inf
1 conc
2 sketch
2 stability

STABLE

$x\uparrow\ +\ \downarrow\ x\downarrow\ -\ \ x\uparrow\ +\ \ x'$

$-\sqrt{5}\qquad\sqrt{5}$



$\frac{d^2x}{dt} = x'' = 2x \cdot x'$

$= 2x(-5+x^2)$

$x''\ -\ \ +\ \ -\ \ +$

$-\sqrt{5}\quad 0\quad \sqrt{5}$

$x\ \frown\ \smile\ \frown\ \smile$

inflection values occur where $x'' = 0$

b. For the rate equation

$$\frac{dx}{dt} = 4 + x^2,$$

find all equilibrium values, determine their stability and sketch graphs of representative solutions $x(t)$ for $t > 0$.

$0 = 4 + x^2$ has no solutions for $x(t)$.

No equilibrium values

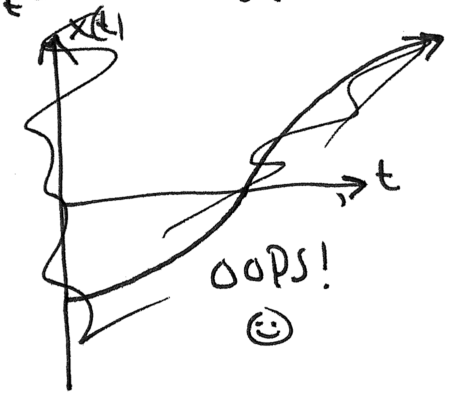
$\frac{dx}{dt} > 0$ so x increases always

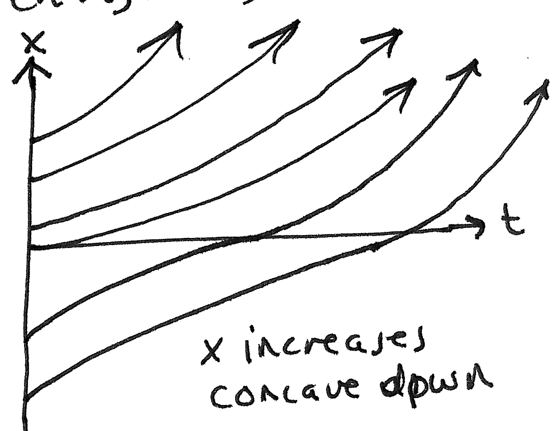
$\frac{d^2x}{dt^2} = 2x \cdot \frac{dx}{dt} = 2x(4+x^2)$ changes sign at $x=0$

$-\ \ +\ \ x''$

0

$\frown\quad\smile$





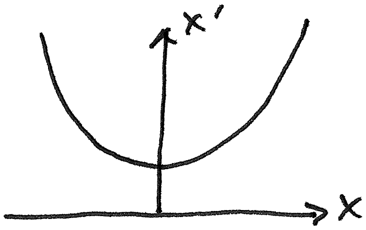
c. The rate equations in (a) and (b) both have the form:

$$\frac{dx}{dt} = r + x^2.$$

The number of equilibrium values depends on the value of r. Find a value of r so that the rate equation only has **one** equilibrium value; determine the stability of this sole equilibrium value and sketch graphs of representative solutions $x(t)$ for $t > 0$.

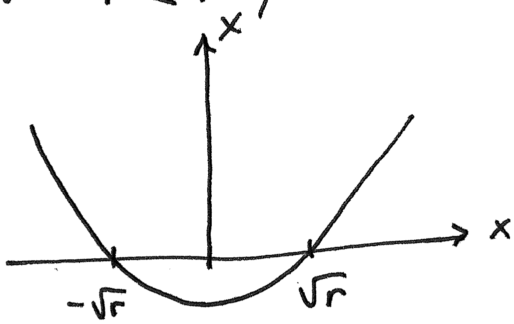
3 $r=0$
1 f'' , inc/dec
1 inf
1 conc
2 sketch
1 stability

When $r > 0$



NO EQUILIBRIUM VALUES

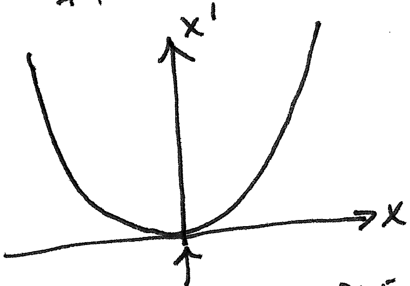
When $r < 0$



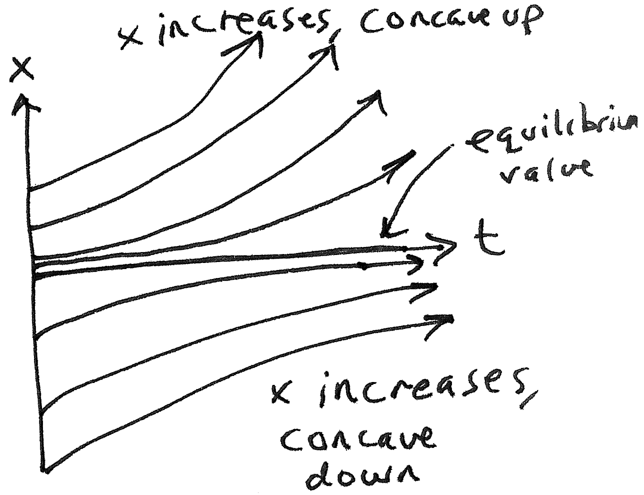
STABLE NOT STABLE

TWO EQUILIBRIUM VALUES

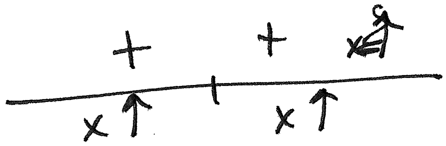
If $r=0$



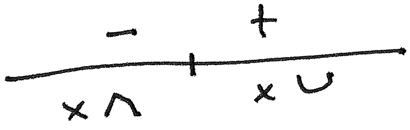
ONE UNSTABLE EQUILIBRIUM VALUE



If $r=0$



$x' = x^2$



$x'' = 2x \cdot x' = 2x^3$

When $r=0$, one unstable equilibrium value of $x=0$, which is also an inflection value.

3. Fun with Numbers. *(19 points.)*
In this problem, your goal is to answer the following question: **"For what positive number is the sum of its reciprocal and four times its square a minimum?"**

a. Give the magical number a name:

Let the magical number be x. 1

b. Write an expression for (i) the reciprocal of the magical number; (ii) four times the square of the magical number; (iii) the sum of (i) and (ii).

(i) $\frac{1}{x}$

(ii) $4x^2$

(iii) $4x^2 + \frac{1}{x}$

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c. Use optimizing techniques to answer the question **in bold** at the top of the page.

Let $\phi(x) = 4x^2 + \frac{1}{x}$, $x > 0$ 2/2 12

$\phi'(x) = 8x - \frac{1}{x^2} = 0$ 4/4

$8x = \frac{1}{x^2}$

$x^3 = \frac{1}{8}$

$x = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ 4/4

2/2 $\phi''(x) = 8 - \frac{-2}{x^3} = 8 + \frac{2}{x^3}$

$\phi''(\frac{1}{2}) = 8 + \frac{2}{(1/2)^3} = 8 + \frac{2}{1/8} = 8 + 2 \cdot 8 = 8 + 16 = 24$

$\phi''(1/2) > 0$ implies $\phi(x)$ has a local minimum at $x = 1/2$. $\phi(\frac{1}{2}) = 4(\frac{1}{2})^2 + \frac{1}{1/2}$

$= 1 + 2 = 3$

4. Your turn to grade! *(30 points.)* In the following limit evaluations, find **any** and **all** mistakes. In the original work, label all errors numerically (i.e. 1, 2, etc.). Then, beneath the work, state why each error is indeed an error. Finally, evaluate the limit correctly. If there is no error, only explain the steps that were taken.

a.

$$\lim_{x\to 0+} \frac{\sin(x)}{x^3} - \frac{1}{x^2} \overset{1}{=} \frac{\sin(x)}{x^3} - \frac{x}{x^3} \overset{2}{=} \lim_{x\to 0+} \frac{\sin(x) - x}{x^3} \overset{3}{=} \lim_{x\to 0+} \frac{\cos(x)}{3x^2} \overset{4}{=} \frac{-\sin(x)}{6x} \overset{5}{=} 0$$

1. The $\lim_{x\to 0^+}$ is missing. The $\frac{1}{x^2}$ has been multiplied by $\frac{x}{x}$ to obtain a common denominator (x^3)
2. The 2 fractions have been placed over the common denominator
3. L'Hôpital's Rule has been applied, INCORRECTLY, should be $\lim_{x\to 0^+} \frac{\cos(x)-}{3x^2}$
   $\overset{L'H}{=} \lim_{x\to 0^+} \frac{-\sin x}{6x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{-\cos(x)}{6} = -\frac{1}{6}$
4. $\lim_{x\to 0^+}$ is missing, L'Hôpital's Rule applied correctly
5. Incorrect answer, should be $-\frac{1}{6}$

b.

$$\lim_{x\to\infty} x^{1/\sqrt{x}} \overset{1}{=} \lim_{x\to\infty} e^{\ln(x^{1/\sqrt{x}})} \overset{2}{=} \lim_{x\to\infty} e^{\frac{1}{\sqrt{x}}\cdot\ln(x)} \overset{3}{=} \lim_{x\to\infty} e^{\frac{\ln(x)}{\sqrt{x}}} \overset{4}{=} e^{\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}} \overset{5}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{0.5x^{-0.5}}} \overset{6}{=} e^{\lim_{x\to\infty}\frac{2x^{0.5}}{x}} \overset{7}{=} e^{\lim_{x\to\infty}\frac{2}{x^{0.5}}} \overset{8}{=} 1$$

1. Property of logs. $e^{\ln A} = A$ for $A > 0$
2. Property of Logs $\ln(B^c) = c\ln B$
3. Algebra: $\frac{1}{B}\cdot A = \frac{A}{B}$
4. Property of limits and continuity of e^x
5. L'Hopital's Rule, correctly applied
6. Algebraic manipulation $\frac{a/b}{c/d} = \frac{a\cdot d}{b\cdot c}$
7. Simplification by cancelling common terms in numerator & denominator $(x^{1/2})$
8. $\lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$ so $e^{\lim_{x\to\infty}\frac{2}{x^{1/2}}} = e^0 = 1$

5. The Multiple Choice You Have Always Wanted. *(20 points.)*
TRUE or FALSE – put your choice in the box. You will receive 4 times more credit for the explanation of your choice than the choice itself. Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE.

(a) **TRUE or FALSE?** For a continuous function $f(x)$ which possesses first derivative $f'(x)$ and second derivative $f''(x)$, whenever $f''(x) = 0$ the graph of the curve $f(x)$ experiences a change in concavity.

$\boxed{\text{FALSE}}$

$+ \quad + \quad f''$
$\cup \quad \cup \quad f$

If $f(x) = x^4$ then $f'(x) = 4x^3$ and $f''(x) = 12x^2$ and $f''(x) = 0$ at $x=0$ but is positive elsewhere, so $f(x)$ does NOT change concavity at $x=0$, where $f''=0$, but only if f'' CHANGES SIGN from $+$ to $-$ or $-$ to $+$.

(b) **TRUE or FALSE?** Every sequence of approximations $x_0, x_1, x_2, \ldots$ generated by using Newton's Method Recursive Algorithm to solve $f(x) = 0$ must converge to a value x^* such that $f(x^*) = 0$ for any initial guess x_0.

$\boxed{\text{FALSE}}$

This is saying Newton's Method ALWAYS converges. Sadly, this is not true.

Newton's Method is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, x_0 given. If $f'(x_0) = 0$, Newton's Method fails instantly.

Also, it is possible for Newton's Method to get caught in an "infinite loop" where it repeats the same two guesses over and over, never approaching x^*.

Also, it is possible for Newton to start "sorta" close and future approx's move away from the exact value instead of towards it.

BONUS *(5 points)*: From the information presented in the following table for an unknown function $\phi(z)$, use Taylor polynomials to obtain your best estimate for $\phi(2)$.

z	$\phi(z)$	$\phi'(z)$	$\phi''(z)$
0	10	10	2

Second order Taylor Polynomial $\phi(a+h) \approx \phi(a) + \phi'(a)h + \frac{\phi''(a)h^2}{2}$

$$\phi(2) \approx \phi(0) + \phi'(0)2 + \frac{\phi''(0)2^2}{2}$$

$$\approx 10 + 10\cdot 2 + 2\cdot\frac{4}{2}$$

$$\approx 10 + 20 + \frac{8}{2}$$

$$\approx 10 + 20 + 4$$

$\boxed{\phi(2) \approx 34}$ (look familiar?)

First order Taylor Polynomial (less accurate)

$$\phi(2) \approx \phi(0) + \phi'(0)2$$

$$\approx 10 + 10\cdot 2$$

$\boxed{\phi(2) \approx 30}$