Math 114 Fall 2005

Report on Exam 2

Ron Buckmire
Angela Gallegos

Point Distribution (N=65)

Range	100+	90+	80+	70+	60+	60-
Grade	A+	A	B	C	D	F
Frequency	7	14	16	12	8	8

Comments

Overall We think that you probably did better on the exam than you thought you did on the way out. The average score was 80. Almost one third of the class earned an A on this second midterm. Please note that the existence of a practice exam does not imply the actual exam will resemble that exam. Practice exam questions are exactly that: questions which can be used to practice your understanding of the concepts and techniques on the upcoming exam. The only thing we can assure you about each exam is that there will be computational, analytical, graphical and verbal components.

#1 The Joke Problem. We hope you enjoyed this problem. Some of those jokes were quite funny! (We may regale you with a few later in the semester.)

#2 The Related Rates Problem. This is a related rate problem similar to those from Homework 18. Since you didn't have calculators we thought that would change your emphasis to being more precise about showing your steps since you don't really want to do all those icky calculation by hand. Most people seemed pretty intent on obtaining a numerical answer with decimal points even though the question doesn't really ask for one. The given expression $12\ln^2(b) - I^{2/3} = constant$ relate $I(x)$ and $b(x)$ at any age x. You are given information at $x = 5$, that the ratio of brain size to IQ is 2, i.e. $b(5)/I(5) = 2$. You are also given $b(5) = 250g$ and brain size is decreasing at a rate of 3 g/year, so $b'(5) = -3$. The question asks how fast is IQ decreasing, i.e. what is $I'(5)$. Using implicit differentiation, the chain rule and the power rule one obtains the expression $12 \cdot 2 \cdot \ln(b) \cdot \frac{1}{b} \cdot \frac{db}{dx} - \frac{2}{3} I^{-1/3} \cdot \frac{dI}{dx} = 0$. Solving for $\frac{dI}{dx}$ produces the expression $\frac{dI}{dx} = \frac{3}{2} \cdot 24 \cdot \frac{\ln(b)}{b}\frac{db}{dx} \cdot I^{1/3}$. At $x = 5$ one knows that $I(5) = b(5)/2 = 125\ g, b'(5) = -3$ and $b(5) = 250\ g$, so $\frac{dI}{dx} = 36 \cdot \frac{\ln(250)}{250} \cdot -3 \cdot (125)^{1/3}$ points/year.

#3 The Limit Problem. There's a reason we did all those limit definition of derivative proofs! **(a)** This question is seeing if you will apply the limit definition of the derivative and combine it with given information about an unknown mystery function to obtain an expression for it's derivative.

$$\begin{aligned}
\Omega'(a) &= \lim_{h\to 0} \frac{\Omega(a+h) - \Omega(a)}{h} \text{ (limit definition of the derivative of } \Omega(x) \text{ at } x = a) \\
&= \lim_{h\to 0} \frac{\Omega(a)\Omega(h) - \Omega(a)}{h} \text{ (Using Property 1 which says that } \Omega(a+b) = \Omega(a)\Omega(b)) \\
&= \lim_{h\to 0} \Omega(a) \cdot \frac{[\Omega(h) - 1]}{h} \text{ (Factoring Terms in order to use Property 2 at the next step)} \\
&= \lim_{h\to 0} \Omega(a) \cdot \lim_{h\to 0} \frac{\Omega(h) - 1]}{h} \text{ (Using Product of Limits equals Limit of the Products)} \\
&= \lim_{h\to 0} \Omega(a) \cdot 1 \text{ (Using Property 2)} \\
&= \Omega(a) \cdot 1 \text{ (Using Property of Limits that Limit of a Constant Function is a Constant)} \\
\Omega'(a) &= \Omega(a)
\end{aligned}$$

(b) So the elementary function must have the fact that its derivative is equal to itself. That's the exponential function. So, $\Omega(x) = e^x$. To test your hypothesis, confirm that it obeys Property 1 and Property 2. $\Omega(a+b) = e^{a+b} = e^a \cdot e^b = \Omega(a) \cdot \Omega(b)$ confirms Property 1. $\lim\limits_{b \to 0} \dfrac{\Omega(b)-1}{b} = \lim\limits_{b \to 0} \dfrac{e^b - 1}{b} = \lim\limits_{b \to 0} \dfrac{e^b}{1} = 1$ using L'Hopital's Rule confirms Property 2. Of course $(e^x)' = e^x$ also confirms the result proven in part (a).

#4 The Verbal Problem. The key to these "argument problems" is to make sure you write something about each statement and clearly state why you agree or disagree with each statement. For statements that you make, you need to provide evidence for how you know the function $f(x) = |x|$ is continuous and where it is or is not differentiable. You also need to be very careful not to make any false statements yourself, like "all continuous functions are differentiable." Note the choice of names Harper, Madison and Sydney. Did you make assumptions about the genders of the students? Why? The basic idea is that $f(x) = |x|$ is continuous everywhere but not differentiable at $x = 0$ because it is not locally linear there. Differentiability of a function at a point implies continuity of a function at that point, and a discontinuity at a point implies a lack of differentiability at a point.

#5 The Computational Problem. This problem is to verify that the rules of differentiation have indeed been completely internalized. **(a)**. You can simplify the function before you differentiate it: $f(t) = 3t^{1/2} + 5t^{-2} + 1 + t^{0.7}e^{4t^2}$ so $f'(t) = 3 \cdot \frac{1}{2}t^{-3/2} - 2 \cdot 5t^{-3} + 0 + 0.7t^{-0.3}e^{4t^2} + t^{0.7}e^{4t^2} \cdot 8t$ using the power rule, derivative of a constant, product rule and chain rule. **(b)** $g(x) = \sin\left(\dfrac{x+1}{x-1}\right) + \cos(\dfrac{3\pi}{2})$, so $g'(x) = \cos\left(\dfrac{x+1}{x-1}\right) \dfrac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2} + 0$ using the quotient rule, chain rule and derivative of a constant. **(c)** $h(u) = u \ln(\sin(\cos(u^{1/3})))$, so $h'(u) = 1 \cdot \ln(\sin(\cos(u^{1/3}))) + u \cdot \frac{1}{\sin(\cos(u^{1/3}))} \cdot \cos(\cos(u^{1/3})) \cdot -\sin(u^{1/3}) \cdot \frac{1}{3}u^{-2/3}$ using the product rule and the chain rule (4 times!)

BONUS To ensure full credit for the graph of $g(x)$ which is the inverse of the $f(x)$ you need to show that you have attempted to preserve the property that the graph of g looks like the graph of f reflected about $y = x$ line so that your sketch and $f(x)$ and the line $y = x$ must all intersect at the same spot (twice). Given that $f(x) = e^x - 3$ then one can find $f^{-1}(x)$ by solving $y = e^x - 3$ for x and then switching x and y. So, $y + 3 = e^x$ and $\ln(y+3) = \ln(e^x) = x$ Thus $g(y) = \ln(y+3)$ or $g(x) = \ln(x+3)$. To confirm that g is indeed the inverse of f one needs to check $f(g(x)) = x$ and $g(f(x)) = x$. $f(g(x)) = f(\ln(x+3)) = e^{\ln(x+3)} - 3 = x + 3 - 3 = x$. $g(f(x)) = g(e^x - 3) = \ln((e^x - 3) + 3) = \ln(e^x) = x\ln(e) = x$. We know $f'(x) = e^x$. According to the inverse function theorem $g'(0) = \dfrac{1}{f'(g(0))} = \dfrac{1}{e^{g(0)}}$. What is $g(0)$? Even without knowing g we know $g(0) = f^{-1}(0)$ =the number x which when plugged into $f(x)$ causes f to output zero. In other words, $0 = e^x - 3 = f(x)$ so $x = \ln(3) = f^{-1}(0)$. This means that $g'(0) = \dfrac{1}{e^{\ln(3)}} = \dfrac{1}{3}$. A more direct way to find $g'(0)$ is since you know $g(x) = \ln(x+3)$, then $g'(x) = \dfrac{1}{x+3}$ and $g'(0) = 1/3$.