Problem 1. (15 points)

Find the exact solution to the following initial value problem. For full credit, show your CHECK that your solution is correct.

$$y' = 4y, \qquad y(3) = 2.$$

Problem 2. (5 points)

Suppose that $g(t)$ is a solution of a rate equation $y' = F(y)$.

Use the Chain Rule to prove that $\phi(t) = g(t-5)$ is also a solution of this rate equation.

Problem 3. (20 points)

The questions below concern the initial value problem:

$$y' = |y|, \qquad y(0) = 5.$$

a) Does the Existence Theorem guarantee that this initial value problem has at least one solution? Explain.

b) Does the Uniqueness Theorem guarantee that this initial value problem has a *unique* solution? Explain.

Problem 4. (20 points)

The following rate equation has unique solutions for all initial conditions:

$$y' = F(y) = (y-2)^2(y-1)$$

a) Find any equilibrium values. Show your work.

b) Classify the equilibrium values you found as asymptotically stable or not asymptotically stable.

c) Find any inflection values. Show your work.

d) Sketch a complete portrait of representative solutions to this rate equation. Be sure to show their concavity correctly.

Problem 5. (15 points)

A function $f(x)$ satisfies the following two conditions:

$$f'(x) = f(x) - 2e^{-x}, \quad -\infty < x < +\infty, \quad \text{and} \quad f(0) = 2.$$

a) Find the second-order Taylor polynomial $P_2(x)$ for $f(x)$ about the point $a = 0$. Show your work.

b) Use $P_2(x)$ to estimate $f(1)$.

Problem 6. (10 points)

Let $R(t)$ denote the rabbit population at time t.

a) In the absence of competition and predation, rabbit populations grow exponentially. Suppose the per capita growth rate is 10 rabbits per year, per rabbit. Write down a rate equation for R which expresses this.

b) Competition causes some rabbits to die who otherwise would not have. Suppose the death rate due to competition is 5 rabbits per year, per possible pair of rabbits. Modify your rate equation in a) to account for death due to competition.

Problem 7. (15 points)

Find the following limits, if they exist, or explain why they do not exist, if they do not. Show your work or explain how you know your answer is correct.

a) $\lim_{x \to 1} \frac{\sin(x-1)}{(x-1)^2}$

b) $\lim_{x \to 0^+} (1/x)^{1/x}$

c) $\lim_{t \to \infty} y(t)$, where $y(t)$ is the solution to the initial value problem

$$y' = (y-2)^2(y-1), \quad y(0) = 3/2. \qquad \textit{Hint: Refer to Problem 4.}$$