Problem 2. (15 points)

Let $g(x) = x \arctan(\ln x)$.

a) Determine the domain and the actual range of g.

$x > 0$ is the domain since the log function is undefined for $x \leq 0$.

b) Use ~~the Squeeze Theorem~~ your calculator to ~~find~~ estimate $\lim_{x \to 0+} x \arctan(\ln x)$.

$$\lim_{x \to 0^+} x \arctan(\ln x) = 0 \cdot \arctan(-\infty) = 0 \cdot \left(-\frac{\pi}{2}\right) = 0$$

x	$x \arctan(\ln x)$
$+0.1$	-0.1161
$+0.01$	-0.01356
$+0.001$	-0.001427
$+0.0001$	-1.46×10^{-4}
10^{-6}	$-.49 \times 10^{-6}$
$\downarrow$ 0	$\downarrow$ 0

c) Use differentiation rules to find $g'(x)$. You may use the fact that $\arctan'(u) = \dfrac{1}{1+u^2}$.

~~$\lim_{x \to 0^+} x \arctan(\ln x) = \lim$~~

Product Rule

$$g'(x) = 1 \cdot \arctan(\ln x) + x \cdot \frac{d}{dx} \arctan(\ln x)$$

$$= \arctan(\ln x) + x \cdot \underbrace{\frac{1}{1+(\ln x)^2} \cdot \frac{1}{x}}_{\text{Chain Rule}}$$

Problem 7. (15 points)

Suppose you know the following:

f and g are inverses of each other
$f(2) = 3$ $\Rightarrow$ $g(3) = 2$
$f'(2) = 4$

a) Which of these two points is on the graph of g? Explain.

$(2, 3)$ $(3, 2)$

If $x = 2$, $y = 3 = f(2)$ this means that $f^{-1}(3) = 2$
When the input to the $f^{-1}(x)$ function is 3, the output is 2.

b) Find the equation of the line tangent to the graph of g at the point you chose in a). Show your work.

$y = g(3) + g'(3)(x-3)$

We know $g(3) = 2$

We also know $g'(x) = \dfrac{1}{f'(g(x))}$

$g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(2)} = \dfrac{1}{4}$

$\boxed{y = 2 + \frac{1}{4}(x-3)}$

Problem 5. (15 points)

The following problems are all related.

a) Find the equation of the line tangent to the graph of $f(x) = \ln(x)$ at the point $(1,0)$.

$f'(x) = \frac{1}{x}$ $f'(1) = 1$

$a = 1$
$f(a) = f(1) = \ln 1 = 0$
$f'(a) = f'(1) = 1$

$y = f(a) + f'(a)(x-a)$

$y = 0 + 1 \cdot (x-1)$

$\boxed{y = x - 1}$

b) Use Taylor's Theorem to complete the following *equation*. (*Note:* You may write the error term as $E(h)$.)

$\ln(1+h) = \ln 1 + h \cdot \frac{1}{1} + E(h)$

$= h + E(h)$

c) Use Taylor's Theorem and the result of b) above to evaluate the following limit:

$$\lim_{h\to 0} \frac{\ln(1+h) - h}{h} = \lim_{h\to 0} \frac{E(h)}{h} = 0$$

Since Relative Error is known to be zero in the limit as $h \to 0$ & $\frac{E(h)}{h}$.

Problem 3. (10 points)

a) Does the function $y(t) = e^{t^2}$ satisfy the following rate equation? Show your work.

$$y'(t) = 2y(t)\sqrt{\ln y(t)}, \quad t \geq 0.$$

$y = e^{t^2}$

$\frac{dy}{dt} = e^{t^2} \cdot 2t \stackrel{?}{=} 2\, y(t) \sqrt{\ln y(t)}$

$\stackrel{?}{=} 2 \cdot e^{t^2} \cdot \sqrt{\ln e^{t^2}}$

$e^{t^2} \cdot 2t \stackrel{?}{=} 2 \cdot e^{t^2} \sqrt{t^2 \ln e} = 2 \cdot e^{t^2} \sqrt{t^2} = 2 \cdot e^{t^2} \cdot t$

Problem 4. (15 points)

b) Imagine a spherical balloon made of a material that can stretch indefinitely without breaking. Water (which is not compressible) is pumped into the balloon at the constant rate of 1 cubic centimeter per minute. At time t, let $V(t)$ be the volume of the balloon and $r(t)$ be its radius. Find the following limits. In some cases, no calculation is necessary (but do briefly explain your reasoning). In other cases, calculation is required.

$V = \frac{4\pi r^3}{3}$

$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

$1 = 4\pi r^2 \frac{dr}{dt}$

$\frac{1}{4\pi r^2} = \frac{dr}{dt}$

$\lim_{t\to\infty} V(t) = \infty$

The balloon increases volume forever

$\lim_{t\to\infty} r(t) = \infty$

The radius of the balloon becomes infinitely large

$\lim_{t\to\infty} r'(t) = 0$

Since as $t \to \infty$, $r \to \infty$ and $r' = \frac{1}{4\pi r^2}$ then $r' \to 0$

(You may use the fact that the volume of a sphere is $V = (4/3)\pi r^3$.)

MATH 110 FALL 2000 EXAM2

1. *(20 points)*. Differentiate the following functions. Do NOT simplify.

a) $p(x) = 3\sqrt{x} + \frac{4}{x} + x - \pi^2$

$p'(x) = 3 \cdot \frac{1}{2\sqrt{x}} - \frac{4}{x^2} + 1 - 0$

(power rule)

b) $q(r) = 2^r \tan(r)$

product rule, exponential rule, trig rule

$q'(r) = 2^r \ln 2 \cdot \tan(r) + 2^r \cdot \sec^2(r)$

c) $r(t) = \frac{e^t}{t^2}$ $= t^{-2} e^t$

quotient rule OR product + chain rule

$r'(t) = \frac{t^2 \cdot e^t - e^t \cdot 2t}{(t^2)^2}$

$r'(t) = -2t^{-3} \cdot e^t + t^{-2} \cdot e^t$

$\frac{t^2 e^t}{t^4} - \frac{e^t 2t}{t^4} = t^{-2} e^t - e^t 2t^{-3}$

d) $h(x) = \ln(\cos(x))$

Chain Rule

$h'(x) = \frac{1}{\cos(x)} \cdot -\sin(x)$

/

MATH 110 FALL 2000 EXAM 2

5. *(15 points).* The following equation is satisfied by points (x, y) lying on a particular curve C:

$$\ln(xy+1) + y^2 = 2^x.$$

a) Find the value(s) of y where the curve C crosses the *positive* y axis.

$y > 0$ and $x = 0$ on the positive y-axis.

$$\ln(0 \cdot y + 1) + y^2 = 2^0$$
$$\ln 1 + y^2 = 1$$
$$y^2 = 1$$
$$y = \pm 1.$$

Since $y > 0$, we want $\boxed{y = 1}$

b) Assuming the curve is locally linear where it crosses the *positive* y axis, find its slope there.

Implicit Differentiation

$$\ln(xy(x)+1) + (y(x))^2 = 2^x$$

$$\frac{1}{xy(x)+1} \cdot [xy(x)+1]' + 2y(x) \cdot y'(x) = 2^x \ln 2$$

$$\frac{1}{xy+1} \cdot [1 \cdot y + x \cdot y'] + 2y \cdot y' = 2^x \ln 2$$

We want to know y' at $(0, 1)$.

$x = 0, y = 1$

$$\frac{1}{0 \cdot 1 + 1} [1 \cdot 1 + 0 \cdot y'] + 2 \cdot 1 \cdot y' = 2^0 \cdot \ln 2$$

$$1 + 2y' = \ln 2$$

$$2y' = \ln 2 - 1$$

$$\boxed{y' = \frac{\ln 2 - 1}{2}}$$