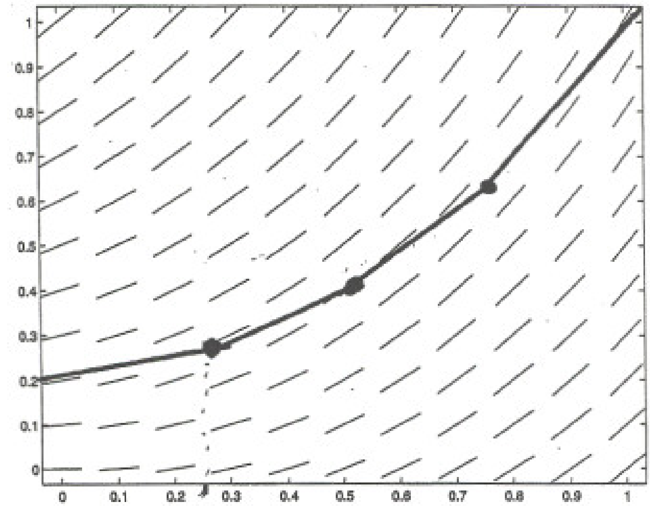
Problem 3. (30 points)

The rate equation $y'(x) = x + y(x)$ has the slope field below.



a) On the slope field above, carefully sketch the graph of the Euler's Method approximation $Y(t)$ to the solution of the initial value problem with initial point $(t_0, y_0) = (0, 0.2)$. Assume that the stepsize is $\Delta x = 0.25$. *Sketch this graph without making any calculations!*

b) It turns out that when using Euler's Method, with stepsize $\Delta x = 0.25$, for the initial value problem in a) above, we find that $Y(0.75) = 0.59375$. Continue using Euler's Method from this point to find $Y(1)$. Show your work.

$$
\begin{aligned}
Y(1) &\approx Y(0.75) + Y'(0.75) \cdot \Delta x \\
&\approx 0.59375 + (0.59375 + 0.75) \cdot 0.25 \\
&\approx 0.9296875
\end{aligned}
$$

c) State *two sources of error* in Euler's Method.

Error due to the assumption of constant slope for a finite (non-zero) step

Error due to calculations occurring on a computer with finite precision

Problem 5. (20 points)

Public health officials are fighting an epidemic of a measles-like disease modeled by the S-I-R model below. A mutant strain breaks out in Smalltown, a community previously free of the disease. This strain has the same transmission and recovery coefficients as the regular strain. However, individuals infected with the mutant strain lose their immunity 50 days after recovering from the illness. The model, revised to reflect this immunity loss, is given below.

$$S' = -.00001\,SI + \frac{1}{50}\,R$$

$$I' = .00001\,SI - \frac{1}{14}\,I$$

$$R' = \frac{1}{14}\,I - \frac{1}{50}\,R$$

a) Clearly explain how and why the S' and R' equations are different from their counterparts in the S-I-R model we studied in class.

S' reflects the increase in the susceptible population due to the "loss of immunity" by $\frac{1}{50}$ of the recovered population per day.
R' reflects the decrease in R as people become susceptible to disease again.

b) In the revised model, how large does S have to be in order for the disease to take hold (i.e. for $I(t)$ to increase for a while)? Show and justify your calculations.

In order for an epidemic to occur (i.e $I(t)$ increase or $I' > 0$) the critical S must be greater than the threshold value, $b/a = \frac{1/14}{0.00001} = \frac{100\,000}{14}$
$= 7142.8$

c) In the revised model, under what conditions does $S(t)$ increase (i.e. $S'(t) > 0$)? Show and justify your calculations.

$S'(t) > 0$ when the number of people being infected per day $(0.00001\,SI)$ is less than the number of people losing their immunity per day $(\frac{1}{50}R)$

Problem 1. (10 points: [4, 4, 2])

Suppose there is a numerical method, called the "Great" Method, for estimating the solution of an initial value problem. You want to use it to estimate the true value of the solution at a particular point.

a) The method has the following property:

"The error (at a given point) is approximately proportional to the *cube* of the stepsize."

Write a mathematical expression which represents this statement.

$$E \propto (\Delta x)^3 \iff E = K(\Delta x)^3$$

b) Use the result of a) to answer this question:

If your error is 0.2 using a stepsize of 0.1, what error do you expect if you use a stepsize of 0.01?

$$0.2 = K(0.1)^3 \Rightarrow \frac{0.2}{0.001} = K \Rightarrow K = 200$$

$$E = 200(\Delta x)^3 \Rightarrow E = 200(0.01)^3 = 200(10^{-6}) = 2 \times 10^{-4} = 0.0002$$

c) What is "great" about this method?

Decrease stepsize by a factor of 10 causes a decrease in error of a factor of $10^3 = 10$

Problem 2. (10 points: [5, 5])

One of the following two statements is FALSE. Identify the FALSE statement and give an example showing it is false.

I. For every linear function, the output is directly proportional to the input.

II. For every linear function, the change in output is directly proportional to the change in input.

1. FALSE. $y = 3x + 2$.
$x = 1, y = 5$
$x = 2, y = 8$

8 is not twice as big as 5 even though 2 is twice as big as 1.

2. TRUE. Every linear function has $\frac{\Delta y}{\Delta x} = m \Rightarrow \Delta y = m\Delta x$

Problem 7. (10 points)

Use Euler's Method with a stepsize $\Delta x = 0.5$ to estimate $y(1.5)$, where $y(x)$ is the exact solution of the intial value problem

$$y'(x) = y(x) + x, \qquad y(0) = 1.$$

Present your answer in the form of a table, with clearly labeled headings.

x	y	y′	Δy
0	1	1	½
½	1½	2	1
1	2½	3½	1¾
1½	4¼	X	X

$\Delta y \approx y' \cdot \Delta t$

$y_{new} \approx y_{old} + \Delta y$

$\boxed{y(1.5) \approx 4.25 \text{ using Euler's Method with } \Delta x = 0}$

Problem 8. (10 points)

Consider the following system of rate equations:

$$A' = A - 0.5\,AB \qquad B' = 0.5\,AB - B.$$

Suppose that $B(t)$ has a maximum value. When B is at this maximum value, what is the value of A? Show your work.

B has maximum $\Rightarrow$ $B' = 0 = 0.5AB - B$

$0 = B(0.5A - 1)$

When B has max value either $B = 0$

or $A = 2$ (i.e $0.5A - 1 = 0$

$0.5A = 1$

$A = 2$)

Problem 1. (20 points)

a) Suppose $f(x) = x^2$ and $g(z) = -z/2$. Evaluate $g(f(4))$.

$f(4) = 4^2 = 16$

$g(f(4)) = g(16) = -\frac{16}{2} = -8$

$\boxed{g(f(4)) = -8}$

b) The gravitational force F between two bodies, as a function of the distance r between them, satisfies

$$F(r) = C/r^2, \qquad \text{where } C \text{ is a positive constant.}$$

What is the *natural mathematical* domain of F?

All possible inputs to $F(r)$ is $\mathbb{R} \setminus \{0\}$, all real numbers excep

What is a *reasonable physical* domain of F?

Distance between bodies must be positive, so $r > 0$ is a reasonable physical domain

c) A linear function passes through the point $(1, 2)$ and has the slope -1. Find the equation for the line which is the graph of this function.

$(x_0, y_0) =$ point $(1,2)$ — slope $(-1) = m$

$y - y_0 = m(x - x_0)$

$y - 2 = -1(x - 1)$

$y - 2 = -x + 1$

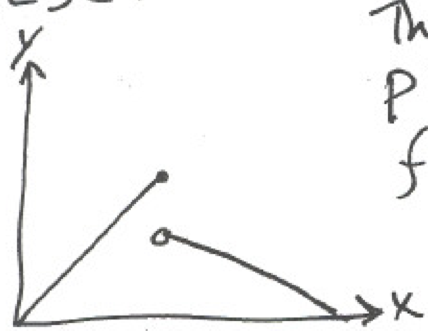
$y = -x + 1 + 2$

$\boxed{y = -x + 3}$

d) TRUE or FALSE: *Every piecewise linear function is continuous.*

If this is true, briefly explain why. It it is false, give a counterexample.

FALSE.



This is an example of a piecewise linear function which is NOT CONTINUOUS

Problem 3. (10 points: [5, 5])

The graph of a continuous piecewise linear function f is defined for all $1 \leq x \leq 3$.

It starts at the point $(1, 2)$,

has the slope -1 on the interval $1 < x < 2$

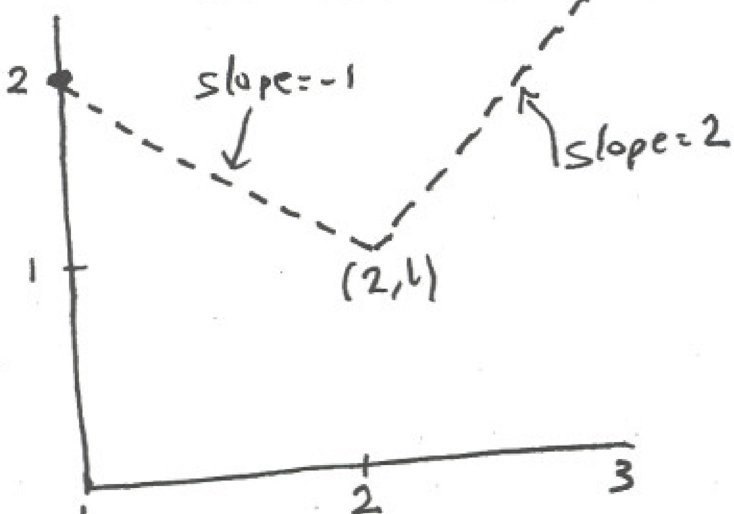
and has the slope $+2$ on the interval. $2 < x < 3$.

Find $f(2)$ and $f(3)$. Show your work.

$f(2) = 1$

$f(3) = 3$

$f(x) = \begin{cases} -1(x-1)+2, & 1<x< \\ 2(x-2)+1, & 2<x \end{cases}$

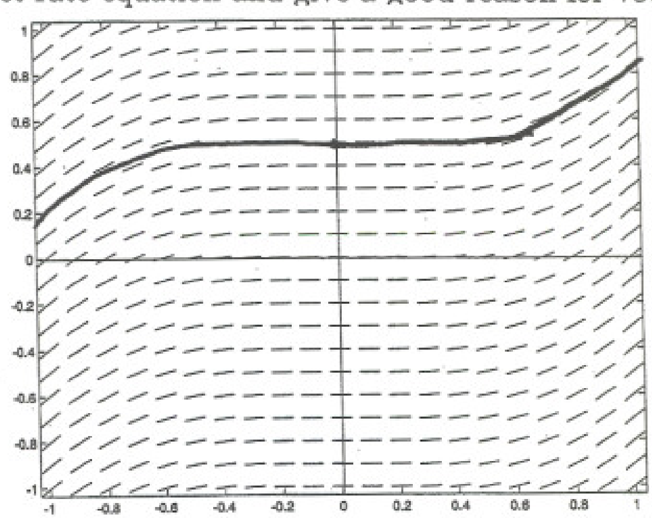


Problem 4. (10 points: [6, 4])

a) A slope field is shown on the following page. It is the slope field for one of the following rate equations:

$$y' = y, \qquad y' = y + x, \qquad y' = x^2$$

Identify the correct rate equation and give a good reason for your choice.



b) On the slope field, sketch the solution to the rate equation that *passes through* the point $(0, \frac{1}{2})$. Show this solution for both negative and positive values of x.