Rate Equations: Equilibrium and Inflection Values

Recall that in the first unit of the course we studied initial value problems consisting of a rate equation and an initial value. We are now going to use what we have learned about first and second derivatives to find important properties of solutions for rate equations of the form

$$y'(t) = F[y(t)], \qquad \text{often written simply as } y' = F(y).$$

Example

$$y' = y \cdot (1 - y)$$

1. For the example above, state what $F(y)$ is.

We will assume that initial value problems for the rate equations we examine have UNIQUE solutions. (This can be shown for the examples above.) This means that one and only one solution graph can pass through any given point (t_0, y_0) in the (t, y) plane.

2. With this assumption, can any solution of $y' = F(y)$ that is NOT a constant function have a local maximum or minimum value? Explain.

3. If $y' = F(y)$, a value y^* of y for which $F(y^*) = 0$ is said to be an **equilibrium value**. If y^* is an equilibrium value for $y' = F(y)$, then $y(t) = y^*$ is a constant solution of the rate equation. Why?

4. Find the equilibrium values for $y' = y \cdot (1 - y)$.

Remember that $y'(t)$ refers to the first derivative of a function $y(t)$ satisfying the rate equation. Since the rate equation give us a formula for $y'(t)$ (it terms of $y(t)$), we can use it to find a formula for $y''(t)$ (again, in terms of $y(t)$):

Example:

$y'(t) = y(t)(1 - y(t))$

$\implies y''(t) = y'(t)(1 - y(t)) + y(t) \cdot (-y'(t)) = y'(t)(1 - 2y(t)) = y(t)(1 - y(t))(1 - 2y(t)),$

since $y'(t) = y(t)(1 - y(t))$.

i.e. $y' = y \cdot (1-y) \implies y'' = y \cdot (1-y) \cdot (1-2y)$

5. Complete the following table for this example:

y	sign of y'	increasing/decreasing	sign of y''	concavity
$y > 1$	$-$	decreasing	$+$	concave up
$y = 1$				
$0.5 < y < 1$				
$y = 0.5$				
$0 < y < 0.5$				
$y = 0$				
$y < 0$				

6. Use the results in your table to sketch solutions to this rate equation. Be sure to include an initial value for each case of y values in the table.

7. If $y' = F(y)$, a value $\hat{y}$ is called an **inflection value** for $y' = F(y)$ if a solution $y(t)$ has an inflection point when it passes through the y-value $\hat{y}$. Find the inflection values for $y' = y \cdot (1-y)$. How do you know that these are inflection values?