Finding Roots and Newton's Method

Finding Roots

We have seen that determining values c where $f'(c) = 0$ are important in finding local maxima and minima for f. Such a value is a *critical point* for f but a *root* for $g = f'$.

Definition: The value r is a **root** of the function g if $g(r) = 0$.

There are many ways to find the roots of a function. One approach which sometimes works is to factor g into factors with known roots. In other cases you may have special knowledge of the function which you can use.

Examples

1. Find the roots of the function $g(x) = (x-3)^4(x+2)$.

2. Find the roots of the function $g(x) = x\ln(x)$.

3. Find the roots of the function $g(x) = \sin(2x)$.

Newton's Method

Newton's Method is a Calculus-based method for approximating roots. In fact, it derives from the Microscope Approximation,

$$\Delta y \approx g'(a)\Delta x.$$

To use this method to approximate a root r for a function g you need several things:

i) an initial guess x_0 *sufficiently close* to the root r,

ii) $g'(r) \neq 0$ on an interval $a < x < b$ containing x_0 and r,

ii) g, g', and g'' continuous on an interval $a < x < b$ containing x_0 and r.

Provided these conditions are satisfied, Newton's Method is *guaranteed* to converge to the root r.

The big question, however, is "HOW CLOSE MUST x_0 BE TO r?" While certain formulas involving the second derivative can be given to address this question, in practice you simply run Newton's Method for a number of iterations to determine whether it is converging to the root you want. If you find it is not doing so, pick another value for x_0. In lab this week you will see examples of how Newton's Method behaves when x_0 is sufficiently close to r, as well as what can happen when x_0 is to far from r.

Deriving Newton's Method from the Microscope Approximation

Suppose we have obtained our nth approximation x_n for the root r of g, and we want to find a better approximation x_{n+1}. Ideally, we would like to know

$$\Delta x = r - x_n.$$

If we knew this exactly, then we could find the root r as $r = x_n + \Delta x$.

Since we don't know Δx exactly, we will appeal to the Microscope Approximation based at our current guess x_n:

$$\Delta y \approx g'(x_n)\Delta x \quad \Longrightarrow \quad \Delta x \approx \frac{\Delta y}{g'(x_n)}.$$

This approximation is valid provided $g'(x_n) \neq 0$. But this will be true, provided x_n is sufficiently close to r, because we know that $g'(r) \neq 0$ and that g' is continuous on an open interval $a < r < b$ containing r.

Although we don't know Δx exactly, we do know Δy exactly! This is because we know $g(x_n)$ and we also know that $g(r) = 0$, so

$$\Delta y = g(r) - g(x_n) = 0 - g(x_n) = -g(x_n).$$

Therefore,

$$\Delta x \approx \frac{\Delta y}{g'(x_n)} = -\frac{g(x_n)}{g'(x_n)}$$

and we choose our next approximation x_{n+1} as

$$x_{n+1} = x_n - g(x_n)/g'(x_n).$$

Together with an initial guess x_0, this recursion defines Newton's Method. Under the conditions listed above, we are guaranteed that

$$\lim_{n\to\infty} x_n = r.$$

Example

4. Let $g(x) = x^2 - 2$. Confirm that g, g', and g'' are continuous (everywhere), and that $g(x) \neq 0$ on an interval containing the root $r = \sqrt{2}$ and the initial guess $x_0 = 1$. Use three iterations of Newton's Method to approximate the root $r = \sqrt{2}$. For greatest accuracy, record your results as fractions or use the memory registers on you calculator to store intermediate results. Compare this with the value for $\sqrt{2}$ given by your calculator.

n	x_n	$\Delta x \approx -g(x_n)/g'(x_n)$
0	1	
1		
2		
3		