Taylor Polynomials, Minima, Maxima and Inflection Points

You are already familiar with the idea of local linearity. If a function f is differentiable at a point a, then as you zoom in on the graph of the function at the point $(a, f(a))$, the graph looks more and more like a straight line. This line is the tangent line, which itself is the graph of the first-degree Taylor polynomial $P_1(x)$ about the point a:

$$P_1(x) = f(a) + f'(a)(x - a).$$

What happens if you now zoom out a little this point? In general, the graph of the function will resemble that of a polynomial of higher degree. The further out from the point that you zoom, the higher will be the degree of the polynomial. We will explore this idea in detail in the next course when we study Taylor polynomials and Taylor series. For now, however, we can use this idea to better understand maxima, minima and inflection points.

1. Assuming that a is a constant, that $f'(a) = 0$ and that $f''(a) \neq 0$, use your knowledge of derivative tests to explain why the quadratic polynomial $P_2(x)$ has either a maximum or a minimum at a:

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

$P_2(x)$ is called the **second-degree Taylor polynomial** for f about a. As you can see, f will have a local max or min at a when $P_2(x)$ has a local max or min at a.

2. Find the second-degree Taylor polynomial $P_2(x)$ about $a = \pi/2$ for $f(x) = \sin(x)$. Graph $P_2(x)$, then classify $a = \pi/2$ as a local max or min for $\sin(x)$.

3. Assuming that a is a constant, that $f''(a) = 0$, and that $f'''(a) \neq 0$, use your knowledge of derivative tests to explain why the cubic polynomial $P_3(x)$ has an inflection point at a:

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f''(a)(x-a)^3, \quad \text{where } 3! = 1 \cdot 2 \cdot 3.$$

$P_3(x)$ is called the **third-degree Taylor polynomial** for f about a. As you can see, f will have an inflection point at a when $P_2(x)$ has an inflection point at a.

4. Find the third-degree Taylor polynomial $P_3(x)$ about $a = 0$ for $f(x) = \sin(x)$. Graph $P_3(x)$. Does $\sin(x)$ have an inflection point at $a = 0$?