## Successive Approximations and Euler's Method

### Why Are Small Stepsizes Good?

As you know, when you apply Euler's Method to an initial value problem using a particular stepsize you get one piecewise linear function. This piecewise linear function is regarded as an approximation to the solution of the initial value problem.

Suppose you then apply Euler's Method to the same initial value problem using a smaller stepsize. This produces another piecewise linear function, interpreted as another approximation to the solution of the initial value problem.

1. From your experience with Euler's Method in this course, what evidence do you have that the approximation produced using the smaller stepsize is generally the better approximation?

2. From your understanding of Euler's Method, WHY do you think that using a smaller stepsize produces a better approximation?

**Successive Approximations Are Better Than A Single Approximation**

You might think that all you need to do with Euler's Method is pick a really small stepsize and use it to produce one piecewise linear approximation to the solution of the initial value problem. However, it is better to repeat Euler's Method several times, each time using a smaller stepsize, to create a sequence of *successive* piecewise linear approximations. Here are some examples to illustrate why this is so.

*You Might Be Missing Something*

The following initial value problem should look familiar:

$$C'(t) = 2t \cdot (C(t))^2, \qquad C(1) = -1.$$

For our last quiz you used Euler's Method with a stepsize of $\Delta t = 1$ to construct a piecewise linear approximation to the solution on the interval $1 \le t \le 3$. On the plot at right, that approximation and two other approximations (using stepsizes $\Delta t = 1/2$ and $\Delta t = 1/4$, respectively) are plotted on this interval. The graph of the actual solution, $C(t) = -1/t^2$, is also plotted. (You'll learn the technique for finding the exact solution to initial value problems like this one in Math 120.)

3. What important feature of the true solution does the approximation using $\Delta t = 1$ fail to capture?

The problem here is not the size of the original $\Delta t$ per se, but the size of that $\Delta t$ relative to how quickly the slope of the true solution changes. Thus, in a problem where the slopes changed very quickly, what might seem to be a small $\Delta t$ might not be small enough.

*How Small Is Small Enough?*

In practice, a sequence of Euler's Method approximations is produced, each with a smaller stepsize, until you notice very little change in approximations when the stepsize is further reduced. An example is given below for another initial value problem with which you are familiar:

$$y'(t) = 2t, \qquad y(0) = 0.$$

The exact solution of this initial value problem is $y(t) = t^2$ (something you will also learn in Math 120).

In the plot at the right, you see what appear to be five function graphs. Actually, there are TEN plotted there. This is a sequence of ten piecewise linear approximations to the solution, each one produced using Euler's Method using a stepsize half as large as before. The last six of these approximations are, however, essentially indistiguishable at this level of resolution. This gives one a lot of confidence that they are quite close to the true solution.

4. What does the *sequence* of approximations tell you in this case that no single one of them can tell you?

*Interlude*

5. Suppose you have two variables $A$ and $B$, and you know that the ratio $A/B$ is constant. Complete the following statement and explain your answer:

   "$A$ is                          to $B$."

**Euler Error at a Point is Approximately Proportional to Stepsize**

We now shift our focus from the entire approximation produced by Euler's Method to an individual point. For example, it can be shown (Math 120 again) that the correct solution to the initial value problem

$$y'(t) = 3t^2, \qquad y(0) = 0$$

has the value 8 when $t = 2$. That is $y(2) = 8$.

What does Euler's Method predict that the value of $y$ will be at $t = 2$? It depends on the stepsize, of course. The table below shows the results of Euler's Method for a sequence of decreasing stepsizes. However, rather than tabulating the estimates for $y(2)$, this table give the values of the ratio ( estimated $y(2) - 8$ )/$\Delta t$.

| $\Delta t$ | $\frac{(\text{ estimated } y(2)-8\,)}{\Delta t}$ |
|---|---|
| $2^1$ | -4 |
| $2^0$ | -5 |
| $2^{-1}$ | -5.5 |
| $2^{-2}$ | -5.75 |
| $2^{-3}$ | -5.875 |
| $2^{-4}$ | -5.9375 |
| $2^{-5}$ | -5.96875 |
| $2^{-6}$ | -5.984375 |
| $2^{-7}$ | -5.9921875 |
| $2^{-8}$ | -5.99609375 |
| $2^{-9}$ | -5.998046875 |

6. What does this table tell you about the relationship between the estimation error and the stepsize as the stepsize gets closer and closer to zero? Why can't you necessarily discover this relationship by looking at only one approximation?