Class 21: Friday October 23

Infinite Sets, Power Sets, Bijections and Cardinality

**Definition: One-To-One Correspondence**

A one-to-one correspondence is a function that operates between two sets where every element of one set is paired with exactly one element of the other set and every element of the other set is paired with exactly one element of the first set. This kind of function is also called a bijection or bijective function.

**Definition: Empty Set**

The empty set is the set which contains no elements. It is often denoted \{\} or \emptyset.

**Definition: Power Set**

The power set of a set is the set of all possible subsets of a set. Note, the empty set is a subset of every set. If the given set is denoted \(A\), then \(P(A)\) is the notation for the power set of \(A\).

**Definition: Cardinality**

The cardinality of a (finite) set is the number of elements in that set. The cardinality of the empty set is zero. The notation for cardinality of a set \(S\) is \(|S|\). So, \(|\{\}| = 0\).

Comparing Finite Sets

We can determine the cardinality of a finite set by placing it into a one-to-one correspondence with a subset of the natural numbers. This is a process that you and I would also call counting!

**Exercise**

What is the cardinality of the following sets?

(a) \{-1, 0, 1, 2, 3, 4, 5, 6, 7\}

(b) \{♥, ♣, ♠, ♦\}

(c) \{one, two, four, six, seven, nine, ten, eleven\}

(d) \{-1, 0, 1, 2\} , \{apple, orange, grape\}, ◊, ♥, □, △

(e) \{-1, 0, 1, 2\} , \{apple, orange, grape\}, \{◊, ♥, □, △\}
The Power Set: The Set Of All Subsets
Defining the power set as the set of all possible subsets of a set produces an interesting result about the cardinality of the power set of sets which all have the same cardinality.

**Group Work**
Let’s find the power sets \(\mathcal{P}(A)\) and \(|\mathcal{P}(A)|\) of the following sets (and their cardinality).

(a) \(A=\{1, 2, 3, 4\}\)

(b) \(B=\{\Box, \bigcirc, \triangle\}\)

(c) \(C=\{\{a, b\}, \{d, e\}\}\)

(d) \(D=\{\spadesuit\}\)

(e) \(E=\{\}\\)
### Theorem: Cardinality of the Power Set

The cardinality of the power set \( \mathcal{P}(A) \) is equal to \( 2^{|A|} \). In other words,

\[
|\mathcal{P}(A)| = 2^{|A|}
\]

### Comparing Infinite Sets: Galileo’s Paradox

Which of the following sets \( A \) or \( B \) has more elements?

- \( A = \{0, 1, 2, 3, 4, \ldots \} = \text{the set containing all Natural Numbers} \)
- \( B = \{0, 1, 4, 9, 16, \ldots \} = \text{the set containing the squares of all Natural Numbers} \)

It is clear that we can place both \( A \) and \( B \) into a 1-to-1 correspondence with each other so that they must have the same number of elements!

Galileo Galilei noticed this issue with infinite sets and so the problem of how to compare infinite sets (especially the two sets listed above) is known as **Galileo’s Paradox**. Galileo believed that the concept of “equal,” “greater than” or “less than” could not be applied to infinite sets precisely for this reason. He put it this way:

“So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes “equal,” “greater,” and “less,” are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.”

### Definition: Aleph Null or \( \aleph_0 \)

The cardinality of the set of the natural numbers is denoted \( \aleph_0 \) and said to be “aleph null” or “aleph zero.” Sets with this cardinality are often said to be **countable** or **countably infinite** or **denumerable**. \( \aleph_0 \) is said to be the first of the **transfinite numbers**.

### Group Work

Consider the following dozen infinite sets comparisons. Write down >, < or = between each pair of sets to indicate their relative size.

<table>
<thead>
<tr>
<th>SET #1</th>
<th>SET #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: {1, 2, 3, 4, \ldots}</td>
<td>[all natural numbers]</td>
</tr>
<tr>
<td>2: {1, 2, 3, 4, \ldots}</td>
<td>[all even natural numbers]</td>
</tr>
<tr>
<td>3: {1, 2, 3, 4, \ldots}</td>
<td>[all odd natural numbers]</td>
</tr>
<tr>
<td>4: {1, 2, 3, 4, \ldots}</td>
<td>[all unit fractions]</td>
</tr>
<tr>
<td>5: {1, 2, 3, 4, \ldots}</td>
<td>[all natural numbers]</td>
</tr>
<tr>
<td>6: [all points on a finite line segment]</td>
<td>[all points on an infinite line]</td>
</tr>
<tr>
<td>7: {1, 3, 5, 7, \ldots}</td>
<td>[all odd natural numbers]</td>
</tr>
<tr>
<td>8: {10, 20, 30, 40, \ldots}</td>
<td>[all multiples of 10]</td>
</tr>
<tr>
<td>9: [all points on an infinite line]</td>
<td>[all points on a line 1&quot; long]</td>
</tr>
<tr>
<td>10: [all points inside a unit circle]</td>
<td>[all points on the circumference of a unit circle]</td>
</tr>
<tr>
<td>11: [all points inside a unit circle]</td>
<td>[all points inside a unit square]</td>
</tr>
<tr>
<td>12: [all points on a line 1/2&quot; long]</td>
<td>[all points on a line 1&quot; long]</td>
</tr>
</tbody>
</table>

### Question:

What is the cardinality of the power set of the natural numbers, i.e. \( |\mathcal{P}(\mathbb{N})| \)?