1. Let $A \subset \mathbb{R}$. Prove rigorously that if a, b, c are real numbers such that $a < b < c$, $a, c \in A$, and $b \notin A$, then A is not connected.
2. (Intermediate Value Theorem) Let X be a connected topological space, $f : X \to \mathbb{R}$ a continuous map. Show that $\forall x, z \in X$, if $f(x) < f(z)$, then $\forall b \in [f(x), f(z)]$, $\exists y \in X$ such that $f(y) = b$.
3. Let $f : I \to \mathbb{R}$ be a continuous map such that $f(0) \geq f(1)$. Prove that f has a **fixed point**, i.e., for some $x \in I$, $f(x) = x$.
4. (Brouwer Fixed Point Theorem, Dimension 1) Let $f : I \to I$ be a continuous map. Prove that f has a fixed point. Hint: Define $g(x) = f(x) - x$, and apply the previous problem to g.
5. Find a continuous map $f : (0, 1) \to (0, 1)$ with no fixed points.
6. Suppose you hike up a trail starting at 8:00 AM on Saturday, camp out Saturday night, and hike back down the same trail, starting at 8:00 AM on Sunday. Prove there is a point on the trail such that you cross it at exactly the same time on both days. Assume that you always stay on the trail. But there are no other assumptions; e.g, you may stop and take a break at different times, or even retrace part of the trail to look for something missing that you think you may have dropped earlier!
7. (Borsuk-Ulam Theorem, Dimension 1) Suppose you have a disk-shaped garden. You build a fence on its circle boundary. The height of the fence is not constant, but is continuous. Prove there is a pair of antipodal (opposite) points on the circle such that the fence has the same height at those two points.
8. Can the floor of a room be so "uneven" that no matter where you put a four-legged table (or stool) on it, it will always wobble? (Assume the legs are straight and vertical, and have the same length; and the floor's surface can be represented by the graph of a continuous function $f(x, y)$ of two real variables.)
9. (Borsuk-Ulam Theorem, Dimension 2)

   Theorem Let $f : S^2 \to \mathbb{R}^2$ be continuous. Then there exist antipodal points $p, -p \in S^2$ such that $f(p) = f(-p)$.

   Use the above theorem (without proof) to show: At any given moment, there exist two antipodal points on the surface of Earth that have the same temperature and pressure.
10. (Brouwer Fixed Point Theorem, Dimension n)

    Theorem Every continuous map $f : \overline{B^n} \to \overline{B^n}$ has a fixed point.

    Use the above theorem (without proof) to show: No matter how you stir a jar of honey, there is some molecule that will end up in the same place it was before the stirring. (We're assuming "continuity in the honey": the closer two molecules are to each other before the stirring, the closer they will be after the stirring.)

Challenge Problems

11. Suppose you have two pancakes of arbitrary shape next to each other on a tray. Prove that with a long enough knife you can, with only one cut, simultaneously divide each pancake into two equal halves (measured by area).
