1. Let $Y = \overline{B_5(0,0)} - B_1(0,0) \subset \mathbb{R}^2$. In the following, prove that $g \sim h$, and $h \approx j$.
   (i) $g : I \to Y$ is defined by $g(t) = (4,0)$.
   (ii) $h : I \to Y$ is defined by $h(t) = (3,0) + (\cos(2\pi t), \sin(2\pi t))$.
   (iii) $j : I \to Y$ is defined by $j(t) = (-3,0) + (\cos(2\pi t), \sin(2\pi t))$.

2. (a) Prove that $\approx$ is an equivalence relation for maps.

   Hint: To prove transitivity, proceed as follows. Let f, g, and h be embeddings of X into Y, such that $f \approx g$ and $g \approx h$. To prove $f \approx h$, we'd like to find a map H from what to what, such that what?

   By definition, $f \approx g$ means there is an isotopy $F : X \times I \to Y$ from f to g. Similarly, there is an isotopy $G : X \times I \to Y$ form g to h.

   First, define a map $J : X \times [0,2] \to Y$ as follows: $J(x,t) = \begin{cases} F(x,t) & \text{if } 0 \le t \le 1 \\ G(x,t-1) & \text{if } 1 < t \le 2 \end{cases}$

   Then let $H(x,t) = J(x,2t)$, and show that H is the desired isotopy.

   Don't forget to also show that $\approx$ is reflexive and symmetric.

   (b) It is also true that $\sim$ is an equivalence relation. In your proof above, what would you need to change, and what would you keep the same, in order to prove this?

Definition Let X be a topological space. A loop whose image is just one point in X is called a **trivial loop**. A loop is said to be **null-homotopic** iff it is homotopic to a trivial loop in X.

(For example, you can probably intuitively see that every loop in $\mathbb{R}^2$ is null-homotopic.)

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3. (a) Prove that any two loops in $\mathbb{R}^2$ are homotopic to each other!

   Hint: Step 1: Show every loop $f : I \to \mathbb{R}^2$ is null-homotopic, as follows. Let $p(t) = (0,0)$ be the trivial loop whose image is the origin. Let $H(x,t) = tf(x,t)$. Explain why $H_0 = p$, and $H_1 = f$.

   Step 2: By the above, $\sim$ is an equivalence relation. Use this together with Step 1 to finish the problem.

   (b) Draw three loops on the torus, T^2, such that no two of them are homotopic to each other. (No proof necessary). Can you find more than three?

4. Recall that $\mathbb{RP}^2$ is defined as (this is one of two definitions we have seen): the closed unit disk with **antipodal** (i.e., opposite) points on its boundary identified; $\mathbb{RP}^2 = D^2/\{\forall x \in \partial D^2,\ x \sim -x\}$. Let $q : D^2 \to \mathbb{RP}^2$ be the quotient map.

   (a) Let A be the horizontal diagonal in D^2, i.e., $A = [-1,1] \times \{0\} \subset D^2$. Let $\alpha = q(A) \subset \mathbb{RP}^2$. Then α is a closed curve in $\mathbb{RP}^2$. Why? Technically, α is not really a loop. Why?

   (b) Give a homeomorphism h from I to A.

   (c) Explain why the composition $q \circ h$ is a loop in $\mathbb{RP}^2$. What is the image of this loop? Do you think this loop is null-homotopic (just Y or N, without proof)?

   (d) Define $g : I \to D^2$ by $g(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $q \circ g$ is a loop in $\mathbb{RP}^2$. Why? Prove that the loop $q \circ g$ is null-homotopic.
