1. Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1$ and at least two of $x, y, z$ are in the set $\{0,1\}$ }. Let $F = \partial(\overline{N_{0.1}(X)})$. Show, using pictures, that $F \simeq nT^2$ for some $n$.

2. Let $(X, d)$ be a metric space, and let $A \subset X$. True or False: $\forall \epsilon > 0,\ N_\epsilon(A) = \bigcup_{a \in A} B_\epsilon(a)$. Prove your answer.

3. Take a strip of paper and glue its two shorter edges in such a way as to create a Möbius band. In the following, let $M = [0,1]^2/\{(0,y) \sim (1, 1-y)\}$ denote the Möbius band, and let $q : [0,1]^2 \to M$ be the corresponding quotient map. Let $A = [0,1] \times \{1/2\} \subset [0,1]^2$.

   (a) Explain why $q(A) \subset M$ is a circle. Draw this circle on the Möbius band that you created. We call this circle the **core** of the Möbius band.

   (b) Use a pair of scissors to cut your Möbius band along its core. Is this a separating circle? Use schematic diagrams to prove your answer.

   (c) Let $B = [0,1] \times \{1/4\} \cup [0,1] \times \{3/4\}$. Is $q(B)$ a circle? Use diagrams to explain.

   (d) Create a new Möbius band, and draw $q(B)$ on it. *Before* cutting along it, draw diagrams to try to predict whether or not $M - q(B)$ is connected. Now cut. Was your prediction correct? Draw diagrams to explain why $q(B)$ does or does not separate $M$.

4. Although it may seem intuitively obvious, the following theorem is actually very difficult to prove rigorously! Use it to prove that every embedded circle in the 2-sphere separates the 2-sphere.

   *Jordan Curve Theorem*: Every embedded circle $C \subset \mathbb{R}^2$ separates $\mathbb{R}^2$.

5. Let $f : X \to Y$ be a homeomorphism between topological spaces. True or false: If $A \subset X$ separates $X$, then $f(A)$ separates $Y$. Prove your answer.

6. Prove, without using Theorem 2 of Section 9, that $\mathbb{R}\mathrm{P}^2 \not\simeq S^2$. Hint: (i) Every circle $C \subset S^2$ separates $S^2$; why? (ii) Show that there exists a non-separating circle $C \subset \mathbb{R}\mathrm{P}^2$ (from above we know that the Möbius band has a non-separating circle). (iii) Assume $\mathbb{R}\mathrm{P}^2 \simeq S^2$, and use (i) and (ii) to get a contradiction.

7. (a) Prove that gluing two Möbius bands along their circle-boundaries yields a Klein bottle: $M \cup_\partial M \simeq K$. (You may or may not find it easier to prove that the Klein bottle can be cut up into two Möbius bands.)

   (b) Prove that the Klein bottle is homeomorphic to the connected sum of two projective planes.

   (c) Use the Extra Credit Problem below to prove that $T^2 \# \mathbb{R}\mathrm{P}^2 \simeq \mathrm{K} \# \mathbb{R}\mathrm{P}^2$.

8. (a) It is possible to travel in $\mathbb{R}^3$ from the point $(-1, 0, 0)$ to the point $(1, 0, 0)$ by walking along straight line segments and without ever touching the $y$-axis. Explain how.

   (b) It is possible to travel in $\mathbb{R}^4$ from the point $(-1, 0, 0, 0)$ to the point $(1, 0, 0, 0)$ by walking along straight line segments and without ever touching the $yz$-plane. Explain how.

Challenge Problem

9. Prove that $T^2 \# \mathbb{R}\mathrm{P}^2 \simeq 3\mathbb{R}\mathrm{P}^2$.