1. (a) Let X and Y be sets, with $A, C \subset X$ and $B, D \subset Y$. Prove $(A \times B) \cap (C \times D)$ is of the form $E \times F$ for some $E \subset X$ and $F \subset Y$.

   (b) Use the above to prove that the product topology indeed satisfies the condition of being closed under finite intersections (as is required by the definition of topology).

2. In $\mathbb{R}^2$ with the standard topology (induced by the Euclidean metric), a subset of the form $(a, b) \times (c, d)$ is called an **open rectangle**.

   (a) Prove that every open rectangle is a union of open balls.

   (b) Prove that every open ball is a union of open rectangles.

3. Prove that $\mathbb{R} \times \mathbb{R}$ (as a product space) is homeomorphic to $\mathbb{R}^2$ (with the topology induced by the Euclidean metric). Hint: Let $\mathcal{T}_1$ be the product topology on $\mathbb{R} \times \mathbb{R}$. Let $\mathcal{T}_2$ be the topology on $\mathbb{R}^2$ induced by the Euclidean metric. We want to show $\mathcal{T}_1 = \mathcal{T}_2$; i.e., a set is open in $\mathbb{R} \times \mathbb{R}$ iff it is open in $\mathbb{R}^2$. Use the previous problem for this.

4. Let $Y = S^1 \subset \mathbb{R}^2$. For each of the following spaces X, determine whether the map $f : X \to Y$ defined by $f(x) = (\cos(2\pi x), \sin(2\pi x))$ is a homeomorphism. Support your answers. (Be careful with checking continuity for f and f^{-1}.)

   (a) $X = [0, 1] \subset \mathbb{R}$.

   (b) $X = [0, 1) \subset \mathbb{R}$.

5. Let X_1 and X_2 be topological spaces. For each $i = 1, 2$, we define the **projection map** $\pi_i : X_1 \times X_2 \to X_i$ by $\pi_i(x_1, x_2) = x_i$. Prove that every projection map is continuous.

   (For example, if $X_1 = X_2 = \mathbb{R}$, then π_1 is the projection map onto the x-axis, while π_2 is the projection map onto the y-axis.)

6. (a) Let C be the quotient space obtained by identifying opposite points on the unit circle; i.e., $C = \{S^1/(x, y) \sim (-x, -y)\}$. It turns out, surprisingly, that C is homeomorphic to S^1. Find a homeomorphism from C to S^1 (or vice versa). You don't have to prove that your map is a homeomorphism.

   (b) Let D be the quotient space $\mathbb{R}/\{x \sim (x + 1)\}$. Which familiar topological space is D homeomorphic to? Find a homeomorphism (without proof) to support your answer.

Challenge Problems

7. Let Y, X_1, and X_2 be topological spaces. For each i, let $f_i : Y \to X_i$ be a given map. Define $f : Y \to X_1 \times X_2$ by $f(y) = (f(x_1), f(x_2))$.

   Prove the Continuity of Multi-component Functions Theorem: f is continuous iff each of its component functions is continuous; i.e., f is continuous iff for each i, f_i is continuous.

8. Let X and Y be topological spaces. Prove that for every point $x \in X$, the subspace $\{x\} \times Y \subset X \times Y$ is homeomorphic to Y.