**Final Exam.** Math 460, Topology.
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Name: ______________________
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Do **only six** of the following problems. 20 points per problem. Closed book. Closed Notes. Only the Definitions-Theorems handout allowed. Please write very legibly.

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1. (a) Let $A \subset \mathbb{R}$. Prove rigorously that if $a, b, c$ are real numbers such that $a < b < c$, $a, c \in A$, and $b \notin A$, then $A$ is not connected.
   (b) (Intermediate Value Theorem) Let $X$ be a connected topological space, $f : X \to \mathbb{R}$ a continuous map. Show that $\forall x, z \in X$, if $f(x) \leq f(z)$, then $\forall b \in [f(x), f(z)]$, $\exists y \in X$ such that $f(y) = b$.

2. In this problem you don't have to prove your maps are continuous or homeomorphisms; just briefly explain your thoughts!
   (a) Give a continuous map $f : (0, 1) \to (0, 1)$ with no fixed points.
   (b) Give a homeomorphism $f : (0, 1) \to (0, 1)$ with no fixed points. (Hint: shift every point to the right in such a way that the closer a point is to 0 or 1, the less it gets shifted.)

3. Prove that $\approx$ ("is isotopic to") is an equivalence relation.

4. (a) Draw (without proof) four loops on a torus so that one of them is null-homotopic, and no two of them are homotopic to each other.
   (b) Find (without proof) an equivalence relation $R$ on $I^2 = [0, 1] \times [0, 1]$ such that $I^2/R$ is homeomorphic to a torus.
   (c) Let $q : I^2 \to T$ be the quotient map that you get from $R$ above. Find (without proof) four paths, $f_i : I \to I^2$, $i = 1, 2, 3, 4$, such that the maps $q \circ f_i$ are loops whose images "roughly look like" your four loops in part (a) above.
   (d) Give a homotopy to show the loop you claimed above is null-homotopic indeed is null-homotopic. (Give the homotopy as a map, not just pictures.)

5. (a) Let $D \subset S^2$ be an open disk of radius $\epsilon$, where $\epsilon$ is some small positive number (say 0.1). Prove that $(S^2 \times S^1) - (D \times S^1)$ is homeomorphic to a solid torus.
   (b) Let $D' \subset S^2$ be another open disk of radius $\epsilon$ such that $\overline{D}$ is disjoint from $\overline{D'}$. Explain (using words and pictures) why $(S^2 \times S^1) - [(D \times S^1) \cup (D' \times S^1)] \simeq T^2 \times I$.

6. (a) Use diagrams, with explanations wherever appropriate, to show that gluing two Möbius bands along their circle boundaries yields a Klein bottle: $M \cup_\partial M \simeq K$.
   (b) Let $D$ be an open disk on a torus $T$. Describe the closed surface obtained by gluing $T - D$ and a Möbius band along their circle boundaries; more precisely, find $n$ such that $(T - D) \cup_\partial M$ is homeomorphic to $nT^2$ or $n\mathbb{R}\mathrm{P}^2$. Explain your reasoning.

7. Let $f : X \to Y$ be a continuous map between topological spaces.
   (a) Show that if $X$ is compact, then $f(X)$ is compact.
   (b) True or false? If $f(X)$ is compact, then $X$ is compact. Prove your answer.

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