Recall definitions: Isotopy, Homotopy, Path, Loop. Also recall:

Definition 1. Let X be a topological space. A loop whose image is just one point in X is called a **trivial loop**. A loop is said to be **null-homotopic** iff it is homotopic to a trivial loop in X.

Example 1. Draw two loops in T^2, one that is null-homotopic, and one that is not.

Definition 2. A topological space is said to be **simply connected** iff every loop in it is null-homotopic.

Example 2. Determine which of the closed surfaces are or are not simply connected.

Example 3.

Q1: Is $\mathbb{R}^3$ simply connected? How about $\mathbb{R}^4$?
Q2: How about $S^2 \times I$?
Q3: How about $S^2 \times S^1$?

Multiplying loops

To every topological space X can we associate a group, called the fundamental group of X. Like compactness, connectedness, and simply connectedness, it is a topological invariant. It is a very important and powerful tool, since it is a bridge between topology and algebra, and allows us to use many of the results we know from group theory! It will take a little work and patience before we get to a precise definition of the fundamental group.

Suppose we have two loops, both of which start and end at the same point $x_0 \in X$. Informally, we can "combine" or "join" these two loops into one loop, as follows: A loop is a one-minute trip ($[0,1]$) which starts and ends at the same point. Start at x_0, travel along the first loop, but go twice as fast as normal, so that after half a minute you will be at x_0 again. Then travel along the second loop, again going twice the normal speed, so that after another half a minute you will be at x_0 . So we just created a new one-minute trip (and it doesn't matter that we visited x_0 in the middle of the trip).

Now let's do this formally. Instead of "combine" or "join", we say *multiply* (because we get a group, as we will soon see). The new loop is called the *product* of the two original loops.

Definition 3. Let $\alpha : I \to X$ and $\beta : I \to X$ be loops, such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = x_0$ for some point $x_0 \in X$. The **product** of α with β, written $\alpha \cdot \beta : I \to X$, is defined as:

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & \text{if} \quad 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if} \quad 1/2 < t \le 1 \end{cases}$$

Example 4. Let $X = \mathbb{R}^2 - \{(0,0),(6,0)\}$. Let α and β be loops that start and end at the point $(3,0)$; the image of α is a circle of radius 3 centered at the origin, and the image of β is a circle of radius 3 centered at $(6,0)$. Both loops are traveled in the counterclockwise direction.

Q: Find precise maps for α and β. [1]

Q: What is the product $\alpha \cdot \beta$? How about $\beta \cdot \alpha$? Are these equal? Are they homotopic? [2]

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[1] $\alpha(t) = 3(\cos(2\pi t), \sin(2\pi t))$; $\beta(t) = (6,0) + 3(-\cos(2\pi t), -\sin(2\pi t))$.
[2] N, Y.

Q: Can two loops that have different initial points be multiplied together? [3]

Let X be a topological space. Let's pick a point $x_0 \in X$. We are going to focus attention only on loops that start and end at x_0.

Definition 4. Let X be a topological space, and x_0 an arbitrary point in X. A loop in X is said to be **based** at x_0 iff its initial and terminal points are x_0. The point x_0 is called the **basepoint** of such a loop.

Definition 5. Let X be a topological space, with $x_0 \in X$. Two loops based at x_0 are said to be **homotopic rel basepoint** if there is a homotopy H between them such that the basepoint remains fixed during the homotopy. In other words, $\forall t \in I$, $H(0, t) = x_0$, and $H(1, t) = x_0$.

Note. Instead of *homotopic rel basepoint* we often just say *homotopic*. Of course we should do this only when it is clear from context whether we mean rel basepoint or not rel basepoint. (*rel* is short for *relative*.)

It is clear from definitions that if two loops are homotopic rel basepoint, then they are homotopic. The converse, however, is not true!

Example 5. In example 4 $\alpha \cdot \beta$ is homotopic to $\beta \cdot \alpha$. Why? But they are not homotopic rel basepoint! Why?

Example 6. Let $f : I \to \mathbb{R}^2$ be given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. We can pictorially represent f by a circle of radius 1 centered at the origin, plus an arrow indicating that the circle is traversed in the counterclockwise direction.

Q: What is the basepoint of f?

Q: Find a loop $g : I \to \mathbb{R}^2$ whose image is the same as the image of f, but it is traversed in the clockwise direction. We think of f and g as *inverses* of each other.

Definition 6. Let $\alpha : I \to X$ be a loop. The **inverse** of α is defined as $\alpha^{-1} : I \to X$, $\alpha^{-1}(t) = \alpha(1 - t)$.

Example 7. Let $A \subset \mathbb{R}^2$ be the annulus bounded by the circles of radius 1 and 3 centered at the origin. Let $f : I \to A$ be given by $f(t) = (2\cos(2\pi t), 2\sin(2\pi t))$.

Q: Find f^{-1}.

Q: Does it *intuitively* seem that $f \cdot f^{-1}$ is null-homotopic?

Q: Prove your answer rigorously.

A group?

Let X be a topological space, with $x_0 \in X$. It seems that the set of all loops based at x_0 might form a group under multiplication as defined above. But to be a group, there are more conditions that we need to check. For example, multiplication needs to be associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

Let's check this. Instead of using the formal definition of product, it is much easier to think in terms of combining one-minute trips. (Note: Although what follows is informal, it does contain the main idea of the argument, and can be made quite precise and rigorous.) First consider the left hand side: $(f \cdot g) \cdot h$. It has two "parts": (1) $(f \cdot g)$; (2) h. Think of $(f \cdot g)$ as one loop, and h as another. So we need to

[3] No; why?

travel each part in half a minute. In order to travel $(f \cdot g)$ in half a minute, how long does it take to travel each of f and g?

Now do a similar analysis on the right hand side: $f \cdot (g \cdot h)$. How long is spent for traveling each of f, g, and h? Are they the same as in the left-hand side?

$\pi_1(X)$

It turns out that $(f \cdot g) \cdot h$ is not *equal* to $f \cdot (g \cdot h)$; but they are homotopic (proved in homework). So multiplication would be associative if we considered homotopic loops to be "equivalent". Recall from homework that "is homotopic to" is an equivalence relation.

Definition 7. Let X be a topological space, with $x_0 \in X$. Let L be the set of all loops based at x_0. The **fundamental group** of X, denoted by $\pi_1(X)$, is the set of all equivalence classes of L under the equivalence relation $\sim$ (which here denotes homotopy *rel basepoint*). The **product** of two equivalence classes $[f], [g] \in \pi_1(X)$ is defined to be $[f \cdot g]$.

To see that $\pi_1(X)$ is really a group, we need to check the following:

- Q1. Is the *product* of two equivalence classes well-defined? (What does "well-defined" mean?)
- Q2. What is the identity element for this multiplication?
- Q3. What is the inverse of an equivalence class?

Remark. The reason we write π_1 is that there are also other groups that we assign to every topological space X, which are denoted by $\pi_2, \pi_3, \cdots$. They are homotopy classes of maps from S^n to X. $\pi_n(X)$ is called the nth homotopy group of X.
