We are finally about to see a precise definition of isotopy. Recall the informal definition: If X_1 and X_2 are subsets of a topological space Y, then X_1 is isotopic to X_2 iff X_1 can be continuously deformed in Y to look like X_2.

Definition 1. Let X and Y be topological spaces. We say an embedding $f : X \to Y$ is **isotopic** to another embedding $g : X \to Y$, denoted $f \approx g$, iff there exists a continuous map $H : X \times I \to Y$ such that $\forall x \in X$

(1) $H(x, 0) = f(x)$.
(2) $H(x, 1) = g(x)$.
(3) $\forall t \in I$, $H(\cdot, t)$ is an embedding of X into Y.

We say H is an **isotopy** from f to g.

Before explaining the notation in condition (3), let's try to see what the "idea" in the above definition is. Think of the map H as a one-minute movie. Time is represented by $t \in I$.

(1) says: at time $t = 0$, we see the embedding f.
(2) says: at the end of the movie, at time $t = 1$, we see the embedding g.
(3) says: at every single instant t during the movie, each frame shows an embedding of X.

One more important feature: In our movie, the closer two frames are temporally, the more similar we want them to look. In other words, as we're watching the movie (the deformation), we should not see any "sudden jumps" in it. Q: Which part of the formal definition corresponds to this feature? [1]

Definition 2. Given a fixed $t \in I$, the map $H(\cdot, t) : X \to Y$ is defined by: $\forall x \in X$, $x \mapsto H(x, t)$.
(Read the symbol "$\mapsto$" as "maps to", or "is sent to".)

Instead of $H(\cdot, t) : X \to Y$ we often write $H_t : X \to Y$. They are equivalent.

Example 1. Let $Y = \mathbb{R}^2$, $X = S^1$. Let's denote the circle of radius r centered at the point $(a, b) \in \mathbb{R}^2$ by $C_r(a, b)$. (So $S^1 = C_1(0, 0)$.)

Q: Find an embedding $f : X \to Y$ whose image is $C_2(0, 0)$. [2]

Q: Find an embedding $g : X \to Y$ whose image is $C_3(7, 0)$.

Q: Is f isotopic to g? To prove your answer, follow the steps below.

Step 1: Find a homeomorphism $h : C_2(0, 0) \to C_3(7, 0)$.

Step 2: For an isotopy, we need to find a map H from what to what?

Step 3: For $\vec{x} = (a, b) \in X$ and $t \in I$, let $H(\vec{x}, t) = (1 - t)f(\vec{x}) + (t)g(\vec{x})$, where f and g are thought of as vector-valued functions. Check to see if this satisfies all three conditions of the definition, for the isotopy we desire.

Theorem 1. $\approx$ is an equivalence relation.

Idea of Proof: We only give an idea why $\approx$ is transitive. You will turn this idea into a rigorous proof in homework! We have a one-minute movie in which f becomes g, and a one-minute movie in which g becomes h. We want a one-minute movie in which f becomes h. First we append the second movie to the end of the first movie. This gives us a two-minute movie in which f becomes h. Then we play this movie at twice the normal speed; so it becomes a one-minute movie, as desired.

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[1] H is continuous in t.
[2] $\forall (a, b) \in S^1$, let $f(a, b) = (2a, 2b)$

Definition 3. An embedded circle in $\mathbb{R}^3$ is called a **knot**. A set of (one or more) disjoint embedded circles in $\mathbb{R}^3$ is called a **link**.

In Knot Theory (a branch of Topology), two knots or links that are isotopic to each other are considered to be equivalent. Any knot that is isotopic to $S^1 \times \{0\} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ is called **the unknot** (also called a trivial knot).

Can you guess what the **unlink** with two components would be defined as? Can you draw a 2-component link that is not isotopic to the unlink?

Paths and loops

Definition 4. Let X be a topological space. A **path** (or a curve) in X is a continuous map $p : I \to X$. A path whose **initial point** $p(0)$ equals its **terminal point** $p(1)$ is called a **loop** (or a closed curve).

Note. A path or a loop is allowed to intersect itself; this is not prohibited by the definition. (So loop $\neq$ scc; loop = closed curve.)

Note. A path or a loop is a *map* from I to X, not just a subset of X! Intuitively, we often think of a path or a loop as the *image* of the map p; this is ok, but only intuitively. It is important to remember, however, that technically it is not just the image of the map, but the map itself we will be working with.

Example 2. Determine whether each of the following is a path, a loop, or neither.

(a) $p : [0,1] \to \mathbb{R}^2$, $p(t) = (t, 2t)$. [3]

(b) $p : [0,1] \to \mathbb{R}^2$, $p(t) = \begin{cases} (t,t) & \text{if } 0 \le t < 1/4 \\ (1/2 - t, 1/4) & \text{if } 1/4 \le t < 1/2 \\ (t - 1/2, 3/4 - t) & \text{if } 1/2 \le t < 3/4 \\ (1-t, 0) & \text{if } 3/4 \le t \le 1 \end{cases}$

(c) $p : [0,1] \to \mathbb{R}^3$, $p(t) = (0,0,0)$. [4]

Homotopy

Definition 5. Let X and Y be topological spaces, and f and g continuous maps from X to Y. A **homotopy** from f to g is a continuous map $H : X \times I \to Y$ such that

(1) $H_0 = f$.
(2) $H_1 = g$.

We say f is **homotopic** to g, and write $f \sim g$. (Note that the only difference between a homotopy and an isotopy is the "third condition": an isotopy is a homotopy in which every H_t is an embedding.)

Example 3. Let $Y = \overline{B_5(0,0)} - B_1(0,0)$ (Y is an annulus). Which of the following loops are homotopic to each other in Y? Which are isotopic to each other? (Remember that a loop is a map; so the definitions of homotopy and isotopy can be applied to loops.)

(a) $f : I \to Y$ is defined by $f(t) = 3(\cos(2\pi t), \sin(2\pi t))$.
(b) $g : I \to Y$ is defined by $g(t) = (4, 0)$.
(c) $h : I \to Y$ is defined by $h(t) = (3,0) + (\cos(2\pi t), \sin(2\pi t))$.
(d) $j : I \to Y$ is defined by $j(t) = (-3,0) + (\cos(2\pi t), \sin(2\pi t))$. [5]

Example 4. T or F: If two maps are isotopic to each other, then they are homotopic to each other. How about the converse? [6]

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[3] Path, not loop.
[4] Path and Loop.
[5] See HW.
[6] Y. N.