Suppose we were 2-dimensional creatures, living in $\mathbb{R}^2$. If Christopher Columbus set out sail, and was somehow able to keep traveling in a straight line, he would never come back home again. On the other hand, if our world were a closed surface (such as ?), then traveling in a "straight line" might eventually get us back home.

Is the same possible for the 3D universe we live in? Is it possible that traveling in a fast spaceship along a "straight line" for a long time might get us back to our starting point? *Can you think of any closed 3-manifolds*?

Review:

Definition 1. For $n \geq 0$, the n**-sphere** is defined as: $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid d(\vec{x}, \vec{0}) = 1\}$.

Theorem 1. For $n \geq m$, S^n cannot be embedded in $\mathbb{R}^m$.

Q: What is the smallest m such that S^2 can be embedded in $\mathbb{R}^m$? [1]

A "flatlander" (a 2-dimensional person living in "flatland", i.e., a 2-dimensional world) cannot really visualize a 2-sphere, since a 2-sphere can be embedded in $\mathbb{R}^3$ but not in $\mathbb{R}^2$. Similarly, we, who live in a world that *locally* looks like $\mathbb{R}^3$, cannot visualize a 3-sphere, even though we might very well be living in one! But we can learn to work with it, as well as with many other things we cannot visualize, by learning from flatlanders!

Example 1. One of several ways a flatlander can think of a 2-sphere is: two closed disks glued along their boundaries. Similarly, we can think of a 3-sphere as two closed 3-balls (i.e., 3-dimensional balls in $\mathbb{R}^3$) glued along their boundaries.

Q: *Closed* has two meanings: one for topological spaces, one for manifolds. Which one do we mean when we say *closed ball*? [2]

Q: Complete the following by defining what $\sim$ should be: $S^3 \simeq [\overline{B_1(0,0,0)} \cup \overline{B_1(5,0,0)}]/\sim$, where $\sim$ is defined by: [3] We sometimes write this as $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$. It means we're gluing the two balls along their boundaries.

Example 2. Let's try to see why the above description of S^3 is consistent with the formal definition given at the beginning. In other words, we'd like to find a homeomorphism between $S^3 = \{\vec{x} \in \mathbb{R}^4 \mid d(\vec{x}, \vec{0}) = 1\}$ and $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$.

Let's first do it in one dimension lower. Here's how a flatlander might describe a homeomorphism between $S^2 = \{\vec{x} \in \mathbb{R}^3 \mid d(\vec{x}, \vec{0}) = 1\}$ and $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$:

(1) Send the North Pole $(0,0,1) \in S^2 \subset \mathbb{R}^3$ to the point $(0,0) \in \overline{B_1(0,0)}$. (2) Send the South Pole $(0,0,-1) \in S^2 \subset \mathbb{R}^3$ to the point $(5,0) \in \overline{B_1(5,0)}$. (3) Send the Equator $\{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1\} \subset S^2 \subset \mathbb{R}^3$ to $\partial(\overline{B_1(0,0)}) = \partial(\overline{B_1(5,0)})$. (4) Send circles in the Northern Hemisphere parallel to the Equator to circles in $\overline{B_1(0,0)}$ centered at $(0,0)$. (5) Send circles in the Southern Hemisphere parallel to the Equator to circles in $\overline{B_1(5,0)}$ centered at $(5,0)$.

Q:What is the intersection of S^2 with the horizontal plane of height $1/2$ in $\mathbb{R}^3$? How about heights 0, 1, -1, 2? These are called horizontal **cross sections** of S^2. [4]

Q: How would you rigorously define a **horizontal hyperplane** in $\mathbb{R}^4$ (i.e., a horizontal $\mathbb{R}^3$ in $\mathbb{R}^4$)? [5]

Q: Now try using horizontal cross sections of S^3 to show it is homeomorphic to $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$.

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[1] 3.

[2] We mean "topological"; i.e., a closed subset of $\mathbb{R}^3$.

[3] $\{(x,y,z) \sim (x+5,y,z)\}$.

[4] S^1, S^1, point, point, ϕ.

[5] $\{(x,y,z,w) \in \mathbb{R}^4 \mid w \text{ is a constant}\}$.

Definition 2. Let X be a topological space. An embedded circle in X is called a **simple closed curve** (scc). (*Simple* means not self-intersecting; *closed* means it's a loop – no endpoints.)

Definition 3. Let X be a topological space, with $A \subset B \subset X$. We say A **bounds** B iff $A = \partial B$. For example, a scc $C \subset X$ is said to *bound* a disk if there exists an embedded closed disk $D \subset X$ such that $C = \partial D$.

Example 3. For 3D beings like us, it is easy to *see* that every scc C in the 2-sphere bounds a disk *on both sides*; i.e., there exist two embedded closed disks $D_1, D_2 \subset S^2$ with disjoint interiors such that $C = \partial D_1 = \partial D_2$. (Although "easy to see", this is rather difficult to prove rigorously. It's called the Jordan Curve Theorem.)

For a flatlander, however, this is not quite as easy to see. Let $C \subset \overline{B_1(0,0)}$ be the circle of radius $1/4$ around the point $(1/2, 0)$. Shade-in each of the two disks that C bounds in $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$.

Example 4. Now repeat the above example in one dimension higher: try to see why every embedded S^2 in S^3 bounds a closed 3-ball on each side.

Example 5. To work with $S^2 \times S^1$, it is often helpful to think of it as $S^2 \times I$ with $S^2 \times \{0\}$ glued to $S^2 \times \{1\}$.

Q: Is $S^2 \times S^1$ a manifold? If so, is it a closed manifold?

Q: Find a 2-sphere in $S^2 \times S^1$ that does not bound a ball on either side.

Q: Find a 2-sphere in $S^2 \times S^1$ that bounds a ball on one side only.

Q: Is there a 2-sphere in $S^2 \times S^1$ that bounds a ball on both sides?

Theorem 2. The only connected 3-manifold in which an embedded 2-sphere bounds a ball on both sides is the 3-sphere.

Proof: (Idea) If an embedded 2-sphere bounds a ball on both sides, then the two balls share the same boundary. But we already saw above that two balls glued along their boundaries yields an S^3.

Theorem 3. $S^3 \not\simeq S^2 \times S^1$.

Proof: Homework.

Example 6. Recall the definition of the connected sum of two n-manifolds, M and N: remove an open n-ball from each manifold; then $M - B_1$ and $N - B_2$ will each have a "new boundary component" homeomorphic to what? [6] Glue these boundaries together to obtain the connected sum of M and N.

Q: What is the connected sum of an arbitrary surface with a 2-sphere? Why?

Q: What is the connected sum of an arbitrary 3-mfd with a 3-sphere? Why?

Example 7. Yet another way of thinking of the 2-sphere and the 3-sphere:

Let B^n denote the closed unit ball in $\mathbb{R}^n$. For example, B^2 is the closed unit disk in $\mathbb{R}^2$.

Q: Is $B^2 \simeq B^1 \times B^1$? Is $B^3 \simeq B^2 \times B^1$?

Q: The boundary of $B^2 \times B^1$ can be viewed as a closed disk glued to a cylinder glued to another closed disk. Do you see how?

Q: The boundary of $B^3 \times B^1$ can be viewed as a ? glued to a ? glued to another ?. This should give us the 3-sphere. Why?

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[6] S^{n-1}.