We have so far seen the classification of all *closed* surfaces. Before talking about orientability, we have one more theorem: the classification of non-closed but compact surfaces.

Theorem 1. Every compact surface (with boundary) is homeomorphic to some closed surface minus a finite set of open disks. In other words, every compact surface can be obtained by removing finitely many open disks from one of S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$.

Sketch of Proof: Let F be a compact surface. Then its boundary consists of a number $m \geq 0$ of circles (why?). Take m disjoint copies $D_1, \cdots, D_m$ of the closed unit disk and glue each of them along its boundary to one of the m circle boundaries of F. The new surface F' we obtain is a closed surface; so, by the classification of closed surface, it is homeomorphic to S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$. So F equals S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$ minus a finite number of open disks.

Orientable vs. non-orientable surfaces

Example 1. Write the symbol $\lfloor$ on a piece of paper. Can you make it look like $\rfloor$ by using only rotations and translations but no reflections? (We are doing "rigid" isotopy, so to speak. This means we can move our $\lfloor$ on the piece of paper freely, but we may not change its shape by any stretching, bending, etc.)

Now write the symbol $\lfloor$ on a Möbius band, and "push" it a full turn along the Möbius band. You must remember that this is a 2-manifold, so ideally it has no thickness; therefore, as you're pushing the symbol along, you should stop as soon as it reaches the starting point, even if it may seem to be on the "other side" of the Möbius band.

If this makes you feel uncomfortable, think of it instead as follows: pretend the symbol $\lfloor$ is *inside* the paper, not *on* it, and it has the same thickness (ideally 0) as the paper itself, so that you can see it equally well from "both sides." Now keep pushing it along, and stop as soon as it returns close to its original copy.

The two copies look different! You've succeeded in making $\lfloor$ look like $\rfloor$, by using only translations and rotations, but no reflections. So, on the Möbius band, $\lfloor$ is isotopic to $\rfloor$.

Now do the same on a cylinder ($S^1 \times [0,1]$). Is $\lfloor$ isotopic to $\rfloor$ on a cylinder?

In mathematics, we express these ideas by saying that the Möbius band is not orientable, while the plane and the cylinder are orientable.

Q: Do you think S^2 is orientable (i.e., $\lfloor$ is *not* isotopic to $\rfloor$ on S^2)? How about a torus?

Definition 1. Let $C = [-1,1] \times \{0\} \cup \{0\} \times [-1,1] \subset \mathbb{R}^2$. ($C$ looks like a small cross.) Let M be a 2-manifold. Given an embedding $h : C \to M$, the **mirror-image** h' of h is defined by $h'(x,y) = h(-x,y)$. We say M is **non-orientable** if there is an embedding $h : C \to M$ which is isotopic to its mirror-image. Otherwise, we say M is **orientable**.

Example 2. Draw a copy of C on a Klein bottle. Label its four endpoints as $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Now draw a mirror-image C' of C (next to the original C), by switching the labels $(1,0)$ and $(-1,0)$, but keeping the other two labels fixed. Are C and C' isotopic?

Note. In the above definition, h and h' have the same image, i.e., $h(C) = h'(C)$. But they are different *maps*; for example, $h(1,0) \neq h'(1,0)$. When we talk about h and h' being isotopic, we are concerned not just with their images, but with the maps themselves. This will become precise when we see the formal definition of *isotopy*, in a future section.

Theorem 2. For all $n \geq 1$, S^2 and nT^2 are orientable, while $n\mathbb{R}\mathrm{P}^2$ is non-orientable.

Proof: Omitted.

Corollary 3. A closed surface is embeddable in $\mathbb{R}^3$ iff it is orientable.

Proof: Homework.

Definition 2. For each $n \geq 1$, the surface nT^2 is said to have **genus** n. S^2 is said to have **genus** 0. ($n\mathbb{R}\mathrm{P}^2$ is said to have **genus** $n/2$.)
