There are lots of different topological spaces, some of which are very strange, counter-intuitive, and pathological (and interesting). Some topological spaces, on the other hand, are in some sense very "nice and intuitive." We are going to focus our studies on some of the "nicer and more intuitive" ones: manifolds. These are in a way the ones that are the least strange and the most useful (or at least the most talked-about). We will need two definitions before defining manifolds.

Definition 1. Given a point x in a topological space X, a **neighborhood** of x is any open set that contains x.

Remark. In some books, a neighborhood of x is defined as any set that contains an open set that contains the point x. Such books would then use the term *open neighborhood* for what we simply call above neighborhood. There are advantages and disadvantages to both approaches.

Example 1. Let $X = [0, 1] \subset \mathbb{R}$. Let's see what are some neighborhoods of say 0.3 in X (not in $\mathbb{R}$). A typical neighborhood is the open interval $(0.2, 0.8)$.

Q: Is $[0, 1]$ a neighborhood of 0.3? Yes. Why?

Q: Is $[0, .7)$ a neighborhood of 0.3? Yes. Why?

Q: Is $(0.2, 0.4) \cup (0.6, 0.7)$ a neighborhood of 0.3? Yes. Why?

Q: Is $[0.2, 0.4)$ a neighborhood of 0.3? No. Why?

Example 2. Let $X = S^1 \subset \mathbb{R}^2$. Give an example (other than S^1 itself) of a neighborhood of the point $(1, 0) \in S^1$. Ans: $\{(x, y) \in S^1 \mid -1/2 < x < 1/2\}$.

Remark. In a metric space, a ball of radius ϵ around a point x is also called an **ϵ-neighborhood** of x.

Definition 2. A topological space X is said to be **locally homeomorphic** to a topological space Y iff every point in X has some neighborhood that is homeomorphic to Y.

Example 3. Let $X = S^1 \subset \mathbb{R}^2$, $Y = (0, 1) \subset \mathbb{R}$.

Q: Is X locally homeomorphic to Y? Yes.

Q: Is $X \simeq Y$? No.

Q: Let $x \in X$. Is *every* neighborhood of x homeomorphic to Y? No; but every point x has *some* neighborhood that is homeomorphic to Y.

Definition 3. An **n-dimensional manifold** (or an n-manifold, for short) is a topological space that is Hausdorff, Second Countable, and locally homeomorphic to an open ball in $\mathbb{R}^n$.

For the time being, you should ignore Hausdorff and Second Countable. The important part of the definition that we'll need to understand is: locally homeomorphic to an open ball in $\mathbb{R}^n$.

Example 4. Is S^1 a 1-manifold? Yes, because it is locally homeomorphic to an open ball in $\mathbb{R}$.

Example 5. Is a torus $S^1 \times S^1$ a manifold? Yes. Every point has some neighborhood that's homeomorphic to an open ball (open disk) in $\mathbb{R}^2$. So it's a 2-dimensional manifold.

Note. The torus is often denoted as T^2.

Example 6. Let $X = x$-axis $\cup$ y-axis. Then $X \subset \mathbb{R}^2$ is a topological space.

Q: Is X a manifold? No: the origin has no neighborhood that's homeomorphic to an open ball.

Q: Is $X - \{(0, 0)\}$ a manifold? Yes. Of what dimension? 1. Is it a connected manifold? No; it has four *components* (we'll see a precise definition of this word later).

Note. Is $(0,1) \simeq \mathbb{R}$? Yes. Is an open ball in $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$? Yes.

Therefore, in the definition of manifold, instead of saying " ... homeomorphic to an open ball in $\mathbb{R}^n$," we can (and often do) just say " ... homeomorphic to $\mathbb{R}^n$."

Example 7. Every point in S^1 has a neighborhood that is homeomorphic to $\mathbb{R}$, so S^1 is a 1-manifold.

Example 8. Q: Is $(0,1) \subset \mathbb{R}^2$ a manifold? Yes, it's a 1-manifold.

Q: How about $[0,1] \subset \mathbb{R}^2$? No, because the endpoints, 0 and 1, do not have neighborhoods that are homeomorphic to an open ball. But $[0,1]$ is a manifold-with-boundary. We'll see a precise definition for this next time.