So far we know of one way to create new topological spaces from known ones: Subspaces. Now we will learn two other methods: 1. Product Spaces; and 2. Quotient Spaces.

Product Spaces

Recall: Given arbitrary sets X, Y, their product is defined as $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$.

Example 1. Let $X = [0,1]$, $Y = [0,1]$. Then $X \times Y$ is called the **closed unit square**.

Example 2. Let $X = \mathbb{R}$, $Y = \mathbb{R}$. Then $X \times Y$ is often denoted $\mathbb{R}^2$.

Example 3. Is every subset of $X \times Y$ of the form $A \times B$, where $A \subset X$ and $B \subset Y$? Ans: No; why?

Definition 1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Their **product** is defined by: A set U is open in $X \times Y$ iff it is a (finite or infinite) union of sets of the form $A \times B$, where A is open in X and B is open in Y. We write: $(X \times Y,\ \mathcal{T})$, where $\mathcal{T} = \{\bigcup A_\alpha \times B_\alpha \mid A_\alpha \in \mathcal{T}_X,\ B_\alpha \in \mathcal{T}_Y\}$.

Example 4. Let $X = [0,1]$, $Y = [0,1]$. What are the open sets in $X \times Y$? Ans: They are unions of "open rectangles."

Q: In the above definition, is $\mathcal{T}$ really a topology on $X \times Y$?

1. Is $\phi \in \mathcal{T}$? Yes; why? Is $X \times Y \in \mathcal{T}$? Yes; why?

2. Is $\mathcal{T}$ closed under arbitrary unions? Yes; why?

3. Is $\mathcal{T}$ closed under finite intersections? $(A \times B) \cap (C \times D) = (? \times ?)$ (HW)

Example 5. What is $S^1 \times [0,1]$? It is called a **cylinder**.

Note. How is the unit circle S^1 a topological space? It is a subspace of $\mathbb{R}^2$.

What is $S^1 \times [0,2]$? It too is a cylinder. Is this cylinder homeomorphic to the one above? Yes; why?

Example 6. What is $S^1 \times S^1$? It is called a **torus**.

Example 7. Let $X = [0,1] \times [0,1]$. We can put a topology on X in two different ways:

Method 1: 1. Start with $\mathbb{R}$, with the Euclidean metric. 2. This induces a topology on $\mathbb{R}$. 3. Then $[0,1]$ inherits the subspace topology from $\mathbb{R}$. 4. Then $[0,1] \times [0,1]$ is given the product topology.

Method 2: $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\} \subset \mathbb{R}^2$; so let X inherit the subspace topology from $\mathbb{R}^2$.

Q: Do these two methods produce the same topology on X? Ans: Yes. Proof: Homework.

We sometimes abbreviate $[0,1] \times [0,1]$ by $[0,1]^2$.

Quotient Spaces (also called Identification Spaces)

If we glue (connect) the two endpoints of a string together, we get a loop. If we "glue" or "identify" the two endpoints of $[0,1]$ into one point, intuitively, we get a circle. If we take two closed disks and glue or identify their two circle boundaries together, we get a sphere S^2. If we glue two opposite edges of a closed square, we get a cylinder. If we pairwise glue all opposite edges of a closed square, we get a torus.

How can we make these ideas precise? We define quotient spaces (or identification spaces).

Recall: Let X be an arbitrary set. An **equivalence relation** on X is a relation (i.e,, a set of ordered pairs) on X that is: **reflexive** ($x \sim x$), **symmetric** (if $x \sim y$, then $y \sim x$), and **transitive** (if $x \sim y$ and $y \sim z$, then $x \sim z$).

For each $x \in X$, the equivalence class of x is defined as: $[x] = \{y \in X \mid y \sim x\}$.

If we take $[0,1]$ and "glue" 0 to 1, we get a circle. We represent the "gluing" by writing $0 \sim 1$.

Definition 2. Let X be a set, $\sim$ an equivalence relation on X, and Q the set of all equivalence classes of $\sim$, i.e, $Q = \{[x] \mid x \in X\}$. Define a map $q : X \to Q$ by: For each $x \in X$, $q(x) = [x]$. The map q is called the **quotient map** (or the *identification map*) from X to Q. We sometimes write $X/\sim$ instead of Q.

Roughly speaking, we give Q the "largest" topology that makes the quotient map q continuous:

Definition 3. Let $(X, \mathcal{T}_X)$ be a topological space, $\sim$ an equivalence relation on X, and $q : X \to Q$ the corresponding quotient map. The **quotient topology** on Q is defined as $\mathcal{T}_Q = \{U \subset Q \mid q^{-1}(U) \in \mathcal{T}_X\}$. In other words, U is declared to be open in Q iff its preimage $q^{-1}(U)$ is open in X. The pair $(Q, \mathcal{T}_Q)$ is called the **quotient space** (or the *identification space*) obtained from $(X, \mathcal{T}_X)$ and the equivalence relation $\sim$.

Example 8. Let $X = [0,1]$. Let $\sim$ be the equivalence relation such that $0 \sim 1$, with no other distinct points being related. We want to understand what points and open sets in Q look like. There is some notation to get used to.

Q: Write the equivalence relation $\sim$ as a set of ordered pairs. Ans: $\{(x,y) \mid (x = y) \text{ or } (x = 0 \text{ and } y = 1) \text{ or } (x = 1 \text{ and } y = 0)\}$.

Q: Write $[0]$ as a set. Ans: By definition, $[0] = \{x \in X \mid x \sim 0\} = \{0,1\}$.

Q: Write $[1]$ as a set. Ans: $[1] = [0] = \{0,1\}$.

Q: Write $[1/2]$ as a set. Ans: $[1/2] = \{1/2\}$.

Q: $q^{-1}([0]) =?$ Ans: $\{0,1\}$. $q^{-1}([1/2]) =?$ Ans: $\{1/2\}$.

Q: Let $A = [0, 1/2)$. Is A open in X? Yes; why? Let $A' = q(A)$. Is A' open in Q? Ans: By definition, A' is open in Q iff its preimage $q^{-1}(A')$ is open in X. What is $q^{-1}(A')$? Be careful: it's not A! $q^{-1}(A') = A \cup \{1\}$; why? So $q^{-1}(A')$ is not open in X; therefore A' is not open in Q.

Q: Let $B = (0, 1/2)$. Is A open in X? Yes; why? Let $B' = q(B)$. Is B' open in Q? Ans: Yes, b/c $q^{-1}(B') = B$, which is open in X.

Q: Find an open subset of Q that contains the point $[0]$. Ans: One trivial answer is Q itself; why?

Q: Find a *proper* open subset of Q that contains the point $[0]$. (Think before reading the answer.) Ans: Let $C = [0, 1/3) \cup (2/3, 1]$. Then $q(C)$ contains $[0]$; why? And $q(C)$ is open in Q; why?

Example 9. Find a homeomorphism from $[0,1]/\{0 \sim 1\}$ to S^1 (just find a map, without proving it is a homeomorphism).

Ans: In polar coordinates: $f(x) = (1, 2\pi x)$. In rectangular coordinates: $f(x) = (\cos(2\pi x), \sin(2\pi x))$.

Example 10. Let X be the unit closed square, $[0,1]^2$. Identify the left edge of X with its right edge: $Y = X/\{(0,b) \sim (1,b)\} \mid b \in [0,1]\}$.

Q: Is X a subset of $\mathbb{R}^2$? Yes. Is Y a subset of $\mathbb{R}^2$? No. Is Y a subset of $\mathbb{R}^3$? No.

Y is an abstract set, with the quotient topology. But Y can be shown to be homeomorphic to the cylinder $S^1 \times [0,1] \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. So we say Y is a cylinder, even though it's not really a subset of $\mathbb{R}^3$.

Example 11. What equivalence relation on $X = [0,1]^2$ gives a torus as the quotient space?

Ans: $[0,1]^2 \,/\, \{(a,0) \sim (a,1)\,,\, (0,b) \sim (1,b), \}$.
