We've seen that $(-1,1) = B_1(0) \subset \mathbb{R}$ is homeomorphic to $\mathbb{R}$. It is also not difficult to prove that the **open unit disk** $B_1(0,0) \subset \mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$. (In dimensions 3 or higher, we say ball; in dimension 2, disk; in dimension 1, interval or segment.)

We also showed that $[-1,1] = \overline{B_1(0)} \subset \mathbb{R}$ is not homeomorphic to $\mathbb{R}$. How? By using connectedness: $[-1,1] - \{1\}$ is connected, but there is no point x for which $\mathbb{R} - \{x\}$ is connected. Similarly, we'd like to show that the **closed unit disk** $\overline{B_1(0,0)} \subset \mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2$. But doing this using connectedness like before is much more difficult. Instead, we learn about another topological invariant: compactness.

Definition 1. Let X be a set, and $A \subset X$. A collection F of subsets of X is called a **cover** of A iff

$$A \subset \bigcup_{B \in F} B$$

"Cover" is both a noun and a verb: F is a cover of A; F covers A.

Example 1. Let $X = \mathbb{R}$, $A = [0,3]$.

Q: Is $F = \{[0,2],[1,5]\}$ a cover of A? Ans: Yes: $A \subset [0,2] \cup [1,5]$.

Q: Is $F = \{[1/n,3] \mid n \in \mathbb{N}\}$ a cover of A? Ans: No: $A \not\subset \bigcup_{B \in F} B$. Why? Since 0 is not in any element of F.

Definition 2. Let X be a set, and $A \subset X$. Let F be a cover of A. A **subcover** of F is a set $F' \subset F$ that covers A.

Example 2. Let $X = \mathbb{R}$, $A = [0,3]$.

Q: Let $F = \{[0,2],[1,5]\}$, $F' = \{[0,2],[1,4]\}$. Is F' a subcover of F? Ans: No, since $F' \not\subset F$.

Q: Let $F = \{[0,2],[1,5]\}$, $F' = \{[0,2]\}$. Is F' a subcover of F? Ans: No, since F' does not cover A.

Q: Let $F = \{[1/n,3] \mid n \in \mathbb{N}\} \cup \{[0,1]\}$. Is F a cover of A? Ans: Yes. Can you find a **finite subcover** of F, i.e., find a cover $F' \subset F$ for A such that F' is finite? Ans: $F' = \{[0,1],[1,3]\}$.

Definition 3. Let X be a topological space, and F a cover of $A \subset X$. F is said to be an **open cover** of A if every element of F is open in X.

Example 3. Let $X = \mathbb{R}$, $A = [0,3]$.

Q: Is $F = \{[0,2],[1,5]\}$ an open cover of A? Ans: No: $[0,2]$ is not open in $\mathbb{R}$ (and neither is $[1,5]$).

Q: Give a finite open cover of A. Ans: $F = \{(-1,2),(1,5)\}$ (or simpler: $F = \{(-1,4)\}$).

Example 4. Let $X = \mathbb{R}$, $A = \mathbb{R}$. Give a finite open cover of A. Ans: $F = \{\mathbb{R}\}$.

Definition 4. A topological space X is **compact** iff every open cover of X has a finite subcover.

Theorem 1. 1. $\mathbb{R}$ is not compact. 2. Every closed interval $[a,b] \subset \mathbb{R}$ is compact.

Proof. Part 2 of this theorem is called the **Heine-Borel Theorem**. Its proof can be found in any standard point set topology book; but since it is somewhat long and involved, we will not do it here.

Proof of part 1: Let $F = \{(n,n+2) \mid n \in \mathbb{Z}\}$. Then clearly F is an open cover of $\mathbb{R}$; but F has no finite subcover, since removing any element $(k,k+2)$ from F causes F to "miss" the point $k+1$. $\square$

Example 5. What is wrong with the following argument?

$F = \{(-\infty, 1)\} \cup \{(n, \infty) \mid n \in \mathbb{Z}\}$ is a cover of $\mathbb{R}$. $F' = \{(-\infty, 1), (0, \infty)\}$ is a finite subcover of F. So $\mathbb{R}$ is compact.

Ans: This cover happened to have a finite subcover, but we didn't prove that *every* cover of $\mathbb{R}$ has a finite subcover.

Example 6. Prove that $(a, b) \subset \mathbb{R}$ is not compact by giving an open cover for it that has no finite subcover.
Ans: $F = \{(a, b - 1/n) \mid n \in \mathbb{N}\}$. Can you prove F has no finite subcover?

Theorem 2. The continuous image of a compact set is compact; i.e., if $f : X \to Y$ is a continuous map between two topological spaces, and if X is compact, then $f(X)$ is compact.

Proof: Homework.

Note. In the above theorem, $f(X) \subset Y$; $f(X)$ may or may not equal Y; Y may or may not be compact.

Corollary 3. Let X and Y be two topological spaces. If $X \simeq Y$, then X is compact iff Y is compact.
