Review

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying: ϕ and X are in $\mathcal{T}$; and $\mathcal{T}$ is closed under unions and finite intersections.

X can be any set; elements of X are called **points**. The pair $(X, \mathcal{T})$ is called a **topological space**. $\mathcal{T}$ is a topology. The elements of $\mathcal{T}$ (they are subsets of X) are called **open** sets.

In other words, a topology on X is a *declaration* of which subsets of X we are *choosing* to call open; we can choose any collection of subsets we desire, as long as the above conditions are satisfied.

Theorem 1. A metric d on a set X induces (in a natural way) a topology $\mathcal{T}$ on X: $A \subset X$ is declared to be"$\mathcal{T}$-open" iff it's "d-open".

Definition 2. Given a topological space $(X, \mathcal{T})$, a subset A of X can be given a topology $\mathcal{T}_A$ by: $V \in \mathcal{T}_A$ iff $V = U \cap A$ for some $U \in \mathcal{T}$. This is called the **subspace topology** (also called the *relative topology*) on A. We say $(A, \mathcal{T}_A)$ is a (topological) **subspace** of $(X, \mathcal{T})$. The topology on A is **induced** by the topology on X.

Definition 3. A function from one topological space to another is **continuous** iff the preimage of every open set is open.

Definition 4. A **homeomorphism** (denoted $\simeq$) is a bijection that is continuous and its inverse is also continuous.

Informal Definition: A **topological invariant** is a property that is preserved by homeomorphisms.

Example 1. We will soon prove the following theorem:

Theorem 2. If $A \simeq B$, then A is connected iff B is connected.

In other words, "connectedness" is preserved by homeomorphisms, so it is a topological invariant.

Connectedness

Intuitively, we'd like to say $[0, 3]$ is connected, while $[0, 1] \cup [2, 3]$ is not.

Definition 5. A topological space X is **connected** iff it is *not* equal to the union of two disjoint nonempty open subsets.

Example 2. Let $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$. (Note: X is implicitly assumed to inherit the subspace topology from $\mathbb{R}$.

Q: Is $[0, 1]$ open in X? Ans: Yes. Why?

Q: Is $[2, 3]$ open in X? Ans: Yes. Why?

Q: Is X connected? Ans: No. Why?

Example 3. Let $X = [0, 1) \cup (1, 2) \cup (2, 3] \subset \mathbb{R}$ (i.e., $X = [0, 3] - \{1, 2\}$).
Q: Is $[0, 1)$ open in X? Ans: Yes. Why?
Q: Is $(1, 2) \cup (2, 3]$ open in X? Ans: Yes. Why?
Q: Is X connected? Ans: No. Why?

Theorem 3. $\mathbb{R}$ is connected.

Proof. (By contradiction.) Suppose $\mathbb{R}$ is not connected. Then, by definition, $\exists A, B \subset \mathbb{R}$ such that $\mathbb{R} = A \cup B$, where A and B are disjoint nonempty open subsets of $\mathbb{R}$. Pick arbitrary points $a \in A$ and

$b \in B$. Let $A' = [a,b] \cap A$. Let $z = \text{lub}(A')$ (A' has a least upper bound because it is bounded and nonempty). We will show that $z \notin A$ and $z \notin B$, which is a contradiction since $z \in \mathbb{R} = A \cup B$.

Claim 1. $z \notin A$.
Proof: By assumption, A is open; so if $z \in A$, then $\exists B_r(z) \subset A$ for some positive r. This implies that for some small enough $\epsilon > 0$, $z + \epsilon$ is in both A and $[a,b]$, which contradicts the fact that z is an upper bound for A'.

Claim 2. $z \notin B$.
Proof: By assumption, B is open, so if $z \in B$, then $\exists B_r(z) \subset B$ for some positive r. This contradicts the fact that z is the *least* upper bound for A', since for some small enough $\epsilon > 0$, $z - \epsilon$ is a smaller upper bound for A'.

□

Theorem 4. $A \subset \mathbb{R}$ is connected iff A is an interval (open, closed, or half open).

Proof: Homework.

Theorem 5. The continuous image of a connected set is connected; i.e, if $f : X \to Y$ is a continuous map between topological spaces, and if X is connected, then f(X) is connected.

Proof: Homework.

Note. $f(X) \subset Y$. $f(X)$ may or may not equal Y. Y may or may not be connected.

Corollary 6. Connectedness is a topological invariant: if X is connected, and Y is homeomorphic to X, then Y is connected. (Equivalently, if $X \simeq Y$, then X and Y are either both connected or both not connected.)

Proof: Homework.

Example 4. Prove $[a,b) \not\simeq (c,d)$.

Sketch of Proof: (By contradiction.) Suppose there exists a homeomorphism $h : [a,b) \to (c,d)$. Let $X = [a,b) - \{a\}$, $Y = (c,d) - \{h(a)\}$. Then it is easy to show that Y is not connected, but X is connected (since $X \simeq \mathbb{R}$). It is also easy to show that the restriction $h|_X : X \to Y$ is a homeomorphism, which implies that Y must be connected. This gives us the desired contradiction.

Q: How would you prove that $[a,b)$ is not homeomorphic to a circle (denoted $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$)?
Ans: we will prove this rigorously later; here is an informal proof: You need to remove at least two points from S^1 to make it disconnected, but you can disconnect $[a,b)$ by removing only one point.

Theorem 7. A topological space X is connected iff it contains no proper subset which is both open and closed in X.

Proof: Homework.

Theorem 8. If A and B are connected subspaces of a topological space X, and if $A \cap B \neq \phi$, then $A \cup B$ is connected.

Proof: Homework.
