The Euclidean Real Line

Recall Calculus (before talking about metric spaces): On the real line $\mathbb{R}$ the distance between two points x and y is given by $|x-y|$. We note that the absolute value function has the following properties:

1. $|x-y| = 0$ iff $x = y$.
2. $|x-y| = |y-x|$.
3. $|x-z| \leq |x-y| + |y-z|$.

Then we define continuity: $|x-a| < \delta$ implies $|f(x)-f(a)| < \epsilon$. Then we notice that we don't need the absolute function itself, but only its three properties listed above in order to prove various (desirable and intuitive) properties of continuous functions, such as: the sum of two continuous functions is continuous; the composition of two continuous functions is continuous; etc.

Generalize to Metric Spaces

We abstract the three properties of the absolute value:

A metric space is a set X with $d : X \times X \to [0, \infty)$ satisfying
1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Then we generalize the definition of continuity: $d(x, a) < \delta \to d(f(x), f(a)) < \epsilon$ (use d instead of absolute value).

A few more definitions to imitate and generalize our familiar Euclidean world:

Ball: $B_r(x) = \{y \in X \mid d(x, y) < r\}$ = the set of all points whose distance from x is less than r .

Open set A (to imitate open intervals in $\mathbb{R}$): $\forall x \in A$, $\exists r$ s.t. $B_r(x) \subset$ the set.

We'd like to generalize even further: talk about open and closed sets and continuous functions without having a distance function! Why do we want to do away with a distance function? The answer, in one word, is: convenience. To elaborate, topology is the study of "deformation-invariant" properties of objects, i.e., properties that remain the same when we distort distances. So the exact values of distances do not matter. It is much easier to talk about open and closed sets, continuity, and many other things, without having to specify a distance function. These comments will become more clear shortly.

We notice an important fact about metric spaces:

Theorem 1. Given metric spaces M and N, a function $f : M \to N$ is continuous iff the preimage (or inverse image) of every open set is open; i.e., for every open set $U \subset N$, $f^{-1}(U)$ is an open subset of M.

Proof: Homework.

This characterization of continuity avoids any mention of a distance function; instead it only talks about open sets. But to define *open*, we still need a distance function, right? No, as we'll see further below!

Important features of open sets in a metric space X :
1. ϕ and X are open.
2. Union of open sets is open.
3. Finite intersection of open sets is open.

Generalize to Topological Spaces

A topology on X is a *declaration* of which subsets of X we are *choosing* to call open; we can choose any collection of subsets we desire, as long as the three conditions listed above are satisfied.

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying:

1. ϕ and X are in $\mathcal{T}$.
2. The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The last two conditions are often stated as: $\mathcal{T}$ is **closed under** unions and finite intersections.

The pair $(X, \mathcal{T})$ is called a **topological space**. The elements of $\mathcal{T}$ are called **open** sets. A subset $A \subset X$ is **closed** iff its complement in X is open.

Example 1. Let $X = \mathbb{R}$, $\mathcal{T} = \{X, \phi, [1,5], (4,6)\}$
Q: Is $\mathcal{T}$ a topology on X ? Ans: No. Why?
Q: What is the "minimum addition" we can make to $\mathcal{T}$ to make it a topology on X ? Ans: Let $\mathcal{T} = \{X, \phi, [1,5], (4,6)\}, (4,5], [1,6)\}$.

Example 2. Let $X = \mathbb{R}$, $\mathcal{T} = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$. Is $\mathcal{T}$ a topology? Ans: No, it is not always closed under infinite unions; e.g., $\bigcup_{n=1}^{\infty}(0,n) = (0,\infty) \notin \mathcal{T}$.

Example 3. Let $X = \mathbb{R}$, $\mathcal{T} = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\} \cup \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty,b) \mid b \in \mathbb{R}\}$. Is $\mathcal{T}$ a topology? Ans: Yes (routine but cumbersome to prove rigorously). Note that $\mathcal{T}$ is the same collection of open sets that we get from the Euclidean metric.

Theorem 2. Every metric space is a topological space. More precisely, given a metric (i.e., a distance function) d on a set X, let $\mathcal{T}$ be the collection of subsets of X that are open according to d. Then $\mathcal{T}$ satisfies the definition of being a topology. $\mathcal{T}$ is said to be the topology **induced** by the metric d.

Proof. Let (X,d) be a metric space, and let $\mathcal{T}$ be the collection of subsets of X that are open according to d. Then, by our previous work (HW), we know that $\mathcal{T}$ satisfies the three conditions in the definition of a topology. Therefore $\mathcal{T}$ is a topology on X. □

Definition 2. Let (X,d) be a metric space. Let $\mathcal{T}$ be the collection of all subsets of X that are open according to the metric d. Then $\mathcal{T}$ is said to be the topology on X **induced** by the metric d.

Example 4. Let X be an arbitrary set, and let d be the discrete metric on X.
Q: What is the topology induced by d, i.e., which subsets of X are open according to d? Ans: $\mathcal{T} = \mathcal{P}(X)$ = the powerset of X (the set of all subsets of X).

Definition 3. For any set X, the **discrete topology** on X is defined as $\mathcal{T} = \mathcal{P}(X)$, i.e., all subsets of X are declared to be open.

Definition 4. For any set X, the **indiscrete topology** on X is defined as $\mathcal{T} = \{\phi, X\}$.

Q: Can you think of a metric d on some set X which induces the indiscrete topology on X?
Ans: If X has only one element, then this is possible. How? If X has two or more elements, then this is impossible (proved in HW).

The theorem above says that every metric space is a topological space. Is every topological space **metrizable** (i.e., there exists a metric which induces the topology)? We just saw that the answer is no (the indiscrete topology on a set with two or more elements). So it's natural to ask: which topological spaces are metrizable? The answer is a "big" theorem which we will *not* go into:
Theorem: A topological space is metrizable iff $\cdots$.

Continuity

Definition 5. Let X and Y be two topological spaces. We say a function $f : X \to Y$ is **continuous** iff for every open set $U \subset Y$, its preimage $f^{-1}(U) \subset X$ is open in X.

Q: Let (X, d_X) and (Y, d_Y) be two metric spaces. So they are also topological spaces. T or F: A function $f : X \to Y$ is "metric-continuous" iff it is "topological-continuous"? Ans: true. Why?

Subspaces

Let X be a topological space, and A a subset of X. Then A inherits a topological structure from X in a natural way:

Definition 6. Let A be a subset of a set X. Given a topology $\mathcal{T}$ on X, we define the **subspace topology** (also called the *relative topology*) on A by $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$. $(A, \mathcal{T}_A)$ is said to be a **subspace** of $(X, \mathcal{T})$. The $\mathcal{T}_A$ is said to be **induced** by $\mathcal{T}$.

Theorem 3. Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. Then the subspace topology on A is a topology.

Proof: Homework.

Example 5. Let $X = \mathbb{R}$ with the standard topology (i.e., the topology induced by the Euclidean metric). Let $A = [2, 5) \subset X$. Then A inherits the subspace topology $\mathcal{T}_A$ from X.

Q: Is the set $[2, 3)$ open, closed, or neither in A? Ans: It is open! Why? Because $[2, 3)$ is the intersection of A with some open set $U \subset X$. Can you find such a U? (There is more than one U that works here.)

To emphasize: $[2, 3)$ is open in A but is not open in X.

Q: Is the set $(2, 3)$ open in A? Ans: Yes. Why?

Q: Is the set $(2, 3]$ open in A? Ans: No. Why?

Topological equivalence, and homeomorphisms

Example 6. Let $A = [0, 1] \subset \mathbb{R}$, $B = [0, 2] \subset \mathbb{R}$ (we give both the subspace topology from $\mathbb{R}$). Intuitively, A and B are equivalent in some sense: stretching A to make it twice as long turns it into B.

There is a precise definition for what topological equivalence means.

Definition 7. Two topological spaces X and Y are **homeomorphic** (or *topologically equivalent*) iff $\exists f : X \to Y$ such that
1. f is 1-1 and onto;
2. f and f^{-1} are both continuous.

When two spaces X and Y are homeomorphic, we write $X \simeq Y$. The function f is called a **homeomorphism** from X to Y.

Continuing our last example...
Q: Can you prove $A \simeq B$?
Ans: Define $f : A \to B$ by $f(x) = 2x$. Why is f a bijection? (Easy.) Why are f and f^{-1} continuous?

Q: Let $A = [0, 2] \subset \mathbb{R}$, $B = [0, 1] \cup [2, 3] \subset \mathbb{R}$. Is A homeomorphic to B? Ans: No. Proof?

It can be very difficult to show that two topological spaces are *not* homeomorphic: If you can't find a homeomorphism, maybe you're just not looking hard enough.

We will soon be able to prove $A \not\simeq B$. Outline of the proof:
Step 1. Prove that A is connected (has only one piece), but B is not (it has two pieces).
Step 2. Prove that connectedness is a *topological invariant*, i.e., it's a property that's preserved under homeomorphisms; i.e., if $A \simeq B$ then either both or neither have that property.