Definition 1. A **metric space** M is a set X and a function $d : X \times X \to [0, \infty)$ such that $\forall x, y, z \in X$
1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ (d is symmetric);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Example 1. $\mathbb{R}$ with the **Euclidean metric** (the "standard" metric):
$X = \mathbb{R}$, $d(x, y) = |x - y|$. Why is this a metric space?

Example 2. $\mathbb{R}$ with the **discrete metric**, denoted $\mathbb{R}_d$:
$X = \mathbb{R}$, $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. Why is this a metric space?

Example 3. $\mathbb{R}^n$ with the **Euclidean metric** :
$X = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times), for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$. Why is this a metric space?

Example 4. $\mathbb{R}^2$ with the **taxicab metric** :
$X = \mathbb{R}^2$, for $a = (a_1, a_2)$, $b = (b_1, b_2)$, $d(a, b) = |a_1 - b_1| + |a_2 - b_2|$. Why is this a metric space?

Note. 1. Unless stated otherwise, whenever we refer to $\mathbb{R}$ as a metric space, we mean "$\mathbb{R}$ with the Euclidean metric."

2. For a metric space $M = (X, d)$, X is called **the underlying set**. We will often abuse notation and write M instead of X, or vice versa; for example, we may write $x \in M$ instead of $x \in X$; or we may refer to X as a metric space, when it's really $M = (X, d)$ that's a metric space.

Definition 2. Given a metric space M, a point $x \in M$, and a real number $r \geq 0$, the **ball** of radius r around x is defined as

$$B_r(x) = \{y \in M \mid d(x, y) < r\}$$

Example 5. In $\mathbb{R}$ with the Euclidean metric, $B_2(1) =$? Ans: The open interval from -1 to 3: $(-1, 3)$.

Example 6. In $\mathbb{R}^2$ with the Euclidean metric, what does $B_2(1, 2)$ look like? (Strictly speaking, we should write $B_2((1, 2))$; but too many parentheses can make in difficult to read, so we slightly abuse notation and write only one set of parenthesis.) How about $B_2(1, 2) \subset \mathbb{R}^3$, what does it look like?

Example 7. In $\mathbb{R}_d$, what is $B_3(8)$? What is $B_{0.5}(8)$? Ans: $B_3(8) = \mathbb{R}$; $B_{0.5}(8) = \{8\}$

Example 8. In $\mathbb{R}^2$ with the taxicab metric, what does $B_1(0, 0)$ look like?

Example 9. Is there a metric on $\mathbb{R}^2$ for which $B_1(0, 0) = (-1, 1) \times (-1, 1)$? Ans: $d(a, b) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$.

Definition 3. A subset A of a metric space M is said to be **open** iff $\forall x \in A$, $\exists r > 0$ such that $B_r(x) \subset A$.

Example 10. The interval $(-1, 1]$ is not open in $\mathbb{R}$. Why? Because there is no positive r for which $B_r(1) \subset (-1, 1]$.

Example 11. The interval $(-1, 1)$ is an open subset of $\mathbb{R}$. Why?

Proof: Given an arbitrary $x \in (-1, 1)$, let $r = \min\{d(x, 1), d(x, -1)\}$. Then $B_r(x) \subset (-1, 1)$, because: Let $y \in B_r(x)$; we'll show $y \in (-1, 1)$. By definition of $B_r(x)$, $d(x, y) < r$; so $d(x, y) < \min\{d(x, 1), d(x, -1)\}$; so $d(x, y) < d(x, 1)$ and $d(x, y) < d(x, -1)$. We want to show $d(0, y) < 1$. By triangle inequality, $d(0, y) \leq d(0, x) + d(x, y)$. So, $d(0, y) < d(0, x) + d(x, 1)$ and $d(0, y) < d(0, x) + d(x, -1)$. If $x \geq 0$, then the right hand side of the first inequality equals 1. If $x < 0$, then the left hand side of the second inequality equals 1. So either way, $d(0, y) < 1$, as desired. We showed that for every $x \in (-1, 1)$, there is a positive r such that $B_r(x) \subset (-1, 1)$. So by the definition of open, $(-1, 1)$ is an open subset of $\mathbb{R}$.

Example 12. Is the interval $(2, \infty)$ open in $\mathbb{R}$? Yes. Why?

Definition 4. A subset A of a metric space M is said to be **closed** iff its complement $A^c = M - A$ is open.

Example 13. $(-\infty, -1] \cup [1, \infty)$ is closed in $\mathbb{R}$. Why?

Example 14. Is $(-\infty, -1]$ closed in $\mathbb{R}$? Yes. Why?

Example 15. Is $[-1, 1]$ closed in $\mathbb{R}$? Yes. Why?

Example 16. $[-1, 1)$ is neither open nor closed in $\mathbb{R}$. Why?

Example 17. $\mathbb{R}$ is open in $\mathbb{R}$. ϕ is open in $\mathbb{R}$. Why?

Example 18. $\mathbb{R}$ is closed in $\mathbb{R}$. ϕ is closed in $\mathbb{R}$. Why?

Example 19. Is $\mathbb{R}$ open or closed or neither in $\mathbb{R}^2$? Ans: closed. Why?

Example 20. Find an open set in $\mathbb{R}_d$. Find a closed set in $\mathbb{R}_d$. Ans: Each of $\mathbb{R}_d$ and ϕ is both open and closed.

(Quote from Munkres's book, *Topology*: Q: "What's the difference between a door and a set?" A: "A door is always either open or closed.")

Note. For emphasis, $B_r(x)$ is sometimes called the *open* ball of radius r around x. In contrast, we have:

Definition 5. The **closed ball** of radius r around x is defined as

$$\overline{B_r(x)} = \{y \in M \mid d(x, y) \leq r\}$$

Example 21. Draw the open and closed balls of radius 5 around the point 2 in $\mathbb{R}$. Draw the open and closed balls of radius 5 around the point $(2, 5)$ in $\mathbb{R}^2$.

Definition 6. Let A be a subset of a metric space M. A point $x \in M$ is said to be a **limit point** of A iff every ball around x contains a point of A other than x.

(Synonyms: cluster point; accumulation point.)

Example 22. Let $M = \mathbb{R}$, $A = [0, 2)$. Which of the points $x = 1, 2, 3$ are limit points of A? Why? Ans: only 1 and 2. What if $A = [0, 1] \cup \{2\}$? Ans: only 1.

Equivalent definition of limit point: x is a limit point of A iff $\forall \epsilon > 0$, $\exists y \in A - \{x\}$ such that $d(x, y) < \epsilon$.

Theorem 1. A subset A of a metric space M is closed iff it contains all its limit points.

Proof. "$\Rightarrow$" : Suppose A is closed. Then, by definition, A^c is open. Let x be a limit point of A. We want to show $x \in A$. By definition of limit point, every open ball around x intersects $A - \{x\}$; therefore no open ball around x is entirely contained in A^c. This implies $x \notin A^c$, since if $x \in A^c$, then there *would be* an open ball around x contained entirely in A^c (since A^c is open). Finally, since $x \notin A^c$, x must be in A, as desired.

"$\Leftarrow$" : (Do yourself!) $\square$

Definition 7. Given a subset A of a metric space M, its **interior** A° is defined as the set of all points $x \in A$ such that some open ball around x is a subset of A.

Example 23. (a) What is the interior of $[2, 5) \subset \mathbb{R}$? Ans: $(2, 5)$.

(b) What is the interior of $(2, 5) \subset \mathbb{R}$? Ans: $(2, 5)$.

(c) What is the interior of the closed ball of radius 2 around the origin in $\mathbb{R}^2$? Ans: the open ball of radius 2 around the origin.

Definition 8. Given a subset A of a metric space M, its **closure** $\overline{A}$ is defined as A union the set of all limit points of A. The **boundary** of A is defined as $\partial A = \overline{A} - A^\circ$.

Example 24. (a) What are the closure and boundary of $[2, 5) \subset \mathbb{R}$? Ans: closure = $[2, 5]$; boundary = $\{2, 5\}$.

(b) What is the closure and boundary of the closed ball of radius 2 around the origin in $\mathbb{R}^2$? Ans: closure = itself; boundary = circle of radius 2 around the origin.

Continuity

For a function f to be continuous roughly means if two point x and y are close, then $f(x)$ and $f(y)$ are close; i.e., if $d(x, y)$ is small, then $d(f(x), f(y))$ is small.

Definition 9. Let M_1, M_2 be metric spaces, with d_1 and d_2 as their corresponding distance functions. A function $f : M_1 \to M_2$ is said to be **continuous at** $a \in M_1$ iff $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_1(a, x) < \delta$ implies $d_2(f(a), f(x)) < \epsilon$. We say f is **continuous** if it is continuous at every point in M_1 .

Example 25. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is continuous.

Proof: Fix an arbitrary point $p \in \mathbb{R}$. We will show f is continuous at p, by showing that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall q \in B_\delta(p)$, $f(q) \in B_\epsilon(f(p))$.

Pick any $\epsilon > 0$. Let $\delta = \epsilon/2$. Then, for any $q \in B_\delta(p)$ we have: $d(f(p), f(q)) = |2p - 2q| = 2|p - q| < 2\delta = \epsilon$. So $f(q) \in B_\epsilon(f(p))$, as desired. Since p was arbitrary, f is continuous at every point in $\mathbb{R}$.

Example 26. Determine whether each of the following functions f and g from $\mathbb{R}$ to $\mathbb{R}$ is continuous at 0. (Support your answers informally, without rigorous proof.)

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$