1. (a) If S^n is defined as the boundary of the closed unit ball in $\mathbb{R}^{n+1}$, then what is S^0?
   (b) The intersection of S^2 with the xy-plane in $\mathbb{R}^3$ is a circle. We call this a **great circle**, since no other circle on S^2 can have a larger radius. Similarly, the intersection of S^2 with any plane through the origin in $\mathbb{R}^3$ is called a great circle. What is the intersection of two great circles?
   (c) Now one dimension higher. Denote points in $\mathbb{R}^4$ by (x, y, z, w). Prove *rigorously* that the intersection of S^3 with the xyz-hyperplane in $\mathbb{R}^4$ is S^2. We call such a 2-sphere a **great sphere**.
   (d) What is the intersection of two great spheres? (Take for example the great sphere cut out by the xyz-hyperplane and the great sphere cut out by the yzw-hyperplane). Prove your answer rigorously.
   (e) Give a definition for a great $(n-1)$-sphere in $S^n \subset \mathbb{R}^{n+1}$. Describe the intersection of two great $(n-1)$-spheres in S^n, and explain your reasoning (it doesn't have to be a rigorous proof, but only a clear and convincing explanation; though a rigorous proof wouldn't be bad either!). Hint: What is the intersection of two $(n-1)$-dimensional hyperplanes that pass through the origin in $\mathbb{R}^n$?

2. A torus T^2 can be defined as a "solid square" I^2 $(= [0, 1] \times [0, 1])$ with its opposite edges identified (with the "right" orientation): $T^2 = I^2/R$, where $R = \{(x, 0) \sim (x, 1)\ ,\ (0, y) \sim (1, y)\}$.
   Similarly, a 3-dimensional torus T^3 can be defined as a solid cube I^3 with its opposite faces identified (with the right orientation). Make this precise by giving an appropriate definition for R': $T^3 = I^3/R'$, where $R' = \cdots$.

3. T^2 can also be defined as $S^1 \times S^1$. We can informally explain how this definition is equivalent to the above definition ($T^2 = I^2/R$) by arguing as follows. For every $t \in I$, the two endpoints of $I \times \{t\} \subset I^2$ are identified into one point; so each $I \times \{t\}$ becomes homeomorphic to $S^1 \times \{t\}$. Therefore, I^2/R is homeomorphic to $S^1 \times I$ with $S^1 \times \{0\}$ identified with $S^1 \times \{1\}$ (with the "right" orientation). Thus, we get $S^1 \times S^1$. Give a similar informal argument to show $I^3/R' \simeq S^1 \times S^1 \times S^1$.

4. (a) What familiar space is a punctured 3-sphere (S^3 minus one point) homeomorphic to? Briefly explain why.
   (b) Let p be an arbitrary point in S^2. Then, in $S^2 \times S^1$, $\{p\} \times S^1$ is a simple closed curve. Draw a schematic picture of this. Call this simple closed curve C. Does C bound a disk in $S^2 \times S^1$?
   (c) What familiar space is $(S^2 \times S^1) - (N_\epsilon(C))^\circ$ (i.e., $(S^2 \times S^1)$ minus the interior of an ϵ-neighborhood of C) homeomorphic to?

5. (a) It is possible to travel in $\mathbb{R}^3$ from the point $(-1, 0, 0)$ to the point $(1, 0, 0)$ by walking along straight line segments and without ever touching the y-axis. Explain how.
   (b) It is possible to travel in $\mathbb{R}^4$ from the point $(-1, 0, 0, 0)$ to the point $(1, 0, 0, 0)$ by walking along straight line segments and without ever touching the yz-plane $\{(x, y, z, w) \in \mathbb{R}^4 \mid x = w = 0\}$. Explain how.

6. A **2-component link** consists of two disjoint circles embedded in $\mathbb{R}^3$. For example, let $X \subset \mathbb{R}^3$ be the unit circle in the xy-plane centered at the origin, and let $Y \subset \mathbb{R}^3$ be the unit circle in the yz-plane centered at $(0, 1, 0)$. Then $X \cup Y$ is a 2-component link; in fact, it has its own name: the **Hopf link** (named after a mathematician). Its two components, X and Y, cannot be "pulled apart"; more precisely, if we let Y' be a unit circle centered at $(0, 5, 0)$, then $X \cup Y$ is not isotopic to $X \cup Y'$. This is not easy to prove rigorously with what we know so far, but should be clear intuitively—do you see it?

Now, $\mathbb{R}^3$ can be viewed as a subset of $\mathbb{R}^4$: $\mathbb{R}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid w = 0\}$. Then $X \cup Y \subset \mathbb{R}^3 \subset \mathbb{R}^4$. Explain informally how $X \cup Y$ can be (isotopically) "pulled apart" in $\mathbb{R}^4$.

Extra Credit Problems

7. (a) Explain how $S^2 \times S^1$ can be viewed as two solid tori glued together along their boundaries.
   (b) Explain how S^3 can be viewed as two solid tori glued together along their boundaries. Hint: view it as the boundary of $B^2 \times B^2$ ($\simeq B^4$).
   (c) Explain why the above does *not* imply $S^3 \simeq S^2 \times S^1$.
8. Give an embedding of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^4$.