Review definitions of isotopy, homotopy, path, loop. Also recall:

Definition 1. Let X be a topological space. A loop whose image is just one point in X is called a **trivial loop**. A loop is said to be **null-homotopic** if it is homotopic as a loop to a trivial loop in X.

Definition 2. Let X be a topological space. X is **path connected** if $\forall x, y \in X$, there is a path $p : I \to X$ from x to y (i.e., $p(0) = x$, $p(1) = y$). X is said to be **simply connected** iff it is path-connected and every loop in it is null-homotopic.

Example 1. Which closed surfaces are simply connected? [1]

Example 2. Is $\mathbb{R}^3$ simply connected? How about $\mathbb{R}^4$? How about $S^2 \times I$? How about $S^2 \times S^1$? [2]

Theorem 1. Suppose X and Y are homeomorphic. Then X is simply connected iff Y is simply connected.

Proof: HW.

Multiplying loops

To every topological space X can we associate a group, called the fundamental group of X. Like compactness, connectedness, and simply connectedness, the fundamental group is a topological invariant. It is a very important and powerful tool, since it is a bridge between topology and algebra. For example, it's not easy to prove that S^3 is not homeomorphic to $S^2 \times S^1$. One way to prove it is to compute the fundamental group of each and show they are different. It will take a bit of work and patience before we get to a precise definition of the fundamental group.

Let's start with an informal discussion. Suppose we have two loops, both of which start and end at the same point $x_0 \in X$. Informally, we can "combine" or "join" these two loops into one loop, as follows: Think of a loop as a one-minute walk ($t \in [0, 1]$) that starts and ends at the same point. Now, to combine the two loops, start at x_0 and walk along the first loop, but at twice the normal speed, so that after half a minute you will be at x_0 again. Then walk along the second loop, again at twice the normal speed, so that after another half a minute you will be at x_0 . So we just created a new one-minute walk (and we don't care that we passed through x_0 in the middle of the walk).

Now let's do this formally. Instead of "combine" or "join", we say *multiply.* The new loop is called the *product* of the two original loops.

Definition 3. Let $\alpha : I \to X$ and $\beta : I \to X$ be loops such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = x_0$ for some point $x_0 \in X$. The **product** of α with β, written $\alpha \cdot \beta : I \to X$, is defined as:

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & \text{if} \quad 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if} \quad 1/2 < t \le 1 \end{cases}$$

Example 3. Let $X = \mathbb{R}^2 - \{(0, 0), (6, 0)\}$. Let α and β be loops that start and end at the point $(3, 0)$ such that the image of α is a circle of radius 3 centered at the origin, and the image of β is a circle of radius 3 centered at $(6, 0)$. Both loops are traveled in the counterclockwise direction.

Q: Find precise maps for α and β. [3] Draw the images of $\alpha \cdot \beta$ and $\beta \cdot \alpha$. Are $\alpha \cdot \beta$ and $\beta \cdot \alpha$ equal as maps? [4] Are $\alpha \cdot \beta$ and $\beta \cdot \alpha$ homotopic as loops? [5]

[1] Only S^2 is.

[2] Y, Y, Y, N.

[3] $\alpha(t) = 3(\cos(2\pi t), \sin(2\pi t))$; $\beta(t) = (6, 0) + 3(-\cos(2\pi t), -\sin(2\pi t))$.

[4] No. Why?

[5] Yes. Proof: HW.

Can two loops that have different initial points be multiplied together? [6]

Let X be a topological space. Let's pick a point $x_0 \in X$. We are going to focus attention on loops that start and end at x_0.

Definition 4. Let X be a topological space, and x_0 an arbitrary point in X. A loop in X is said to be **based** at x_0 if its initial and terminal points are x_0. The point x_0 is called the **basepoint** of such a loop.

Definition 5. Let X be a topological space. Two paths starting at x_0 and ending at x_1 are said to be **homotopic rel endpoints** or **path homotopic** if there is a homotopy H between them such that the endpoints remain fixed during the homotopy. In other words, $\forall t \in I$, $H(0,t) = x_0$, and $H(1,t) = x_1$. If α and β are path homotopic, we write $\alpha \sim_{\mathrm{p}} \beta$.

It is clear from this definition that if two loops are homotopic rel basepoint, then they are homotopic. The converse, however, is not true!

Example 4. In example 3, $\alpha \cdot \beta$ and $\beta \cdot \alpha$ are homotopic as loops. But they are not homotopic rel basepoint (this is not easy to prove, but, hopefully, it is at least intuitively plausible to you).

Example 5. Let $f : I \to \mathbb{R}^2$ be given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. We can pictorially represent f by a circle of radius 1 centered at the origin, plus an arrow indicating that the circle is traversed in the counterclockwise direction.

Q: What is the basepoint of f? [7]

Q: Find a loop $g : I \to \mathbb{R}^2$ whose image is the same as the image of f, but it is traversed in the clockwise direction. We think of f and g as *inverses* of each other.

Definition 6. Let $\alpha : I \to X$ be a loop. The **reverse** of α is defined as $\bar{\alpha} : I \to X$, $\bar{\alpha}(t) = \alpha(1-t)$.

(The reverse of α is not the same as the inverse of α; $\bar{\alpha} \neq \alpha^{-1}$.)

Example 6. Let $A \subset \mathbb{R}^2$ be the annulus bounded by the circles of radius 1 and 3 centered at the origin. Let $f : I \to A$ be given by $f(t) = (2\cos(2\pi t), 2\sin(2\pi t))$. Find $\bar{f}$. Try to see (intuitively) why $f \cdot \bar{f}$ is null-homotopic.

A group?

Let X be a topological space, with $x_0 \in X$. It seems that the set of all loops based at x_0 might form a group under multiplication as defined above. But to be a group, there are a few conditions that we need to check. For example, multiplication needs to be associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

Let's check this. Instead of using the formal definition of product, it is much easier to think in terms of combining one-minute walks. (Note: Although what follows is informal, it does contain the main idea of the argument, and can be made rigorous.) First consider the left hand side of $(f \cdot g) \cdot h = f \cdot (g \cdot h)$. Think of $(f \cdot g)$ as one loop, and h as another. So we need to travel each part in half a minute. If it takes half a minute to travel $(f \cdot g)$, how long does it take to travel each of f and g? [8]

Now do a similar analysis on the right hand side: $f \cdot (g \cdot h)$. How long is spent for traveling each of f, g, and h? [9] So is the left-hand side equal to the right hand side, i.e, are they equal as *maps*? [10]

[6] No; why?
[7] $(1, 0)$.
[8] Quarter minute.
[9] Half, quarter, quarter minute.
[10] No.

Although $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are not equal, they are homotopic as loops (proved in homework). So multiplication would be associative if we considered homotopic loops to be "equivalent." Recall from homework that "is homotopic to" is an equivalence relation. One can similarly prove that "being homotopic as loops" is also an equivalence relation.

Definition 7. Let X be a path-connected topological space, with $x_0 \in X$. Let L be the set of all loops based at x_0. The **fundamental group** of X, denoted by $\pi_1(X)$, is the set of all equivalence classes of L under the equivalence relation $\sim_\mathrm{p}$. The **product** of two equivalence classes $[f], [g] \in \pi_1(X)$ is defined to be $[f \cdot g]$.

To prove that $\pi_1(X)$ is a group, we need to verify the following:

- The product of two equivalence classes well-defined.

  Q: What does "well-defined" mean here? [11]

- There is an identity element for this multiplication. (What is it?)
- The (multiplicative) inverse of the equivalence class of a loop is the equivalence class of the reverse of the loop.
- If we pick a different base point $x_1 \neq x_0$, we get the same fundamental group, up to group isomorphism.

Remark. The reason we write π_1 is that there are also other groups that we assign to every topological space X, which are denoted by $\pi_2, \pi_3, \cdots$. They are homotopy classes of maps from S^n into X; $\pi_n(X)$ is called the nth homotopy group of X.

Finally, it can be shown that $\pi_1(S^2 \times S^1) = \mathbb{Z}$ and $\pi_1(S^3) = \{1\}$ (the trivial group). Hence $S^2 \times S^1 \not\cong S^3$.

[11] It means: if $f_1 \sim_\mathrm{p} f_2$ and $g_1 \sim_\mathrm{p} g_2$, then does it follow that $[f_1 \cdot g_1] \sim_\mathrm{p} [f_2 \cdot g_2]$?