Review

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying: ϕ and X are in $\mathcal{T}$; and $\mathcal{T}$ is closed under unions and finite intersections. X can be any set; elements of X are called **points**. The pair $(X, \mathcal{T})$ is called a **topological space**. $\mathcal{T}$ is a topology. The elements of $\mathcal{T}$ (they are subsets of X) are called **open** sets. A topology on X is a *declaration* of which subsets of X we are *choosing* to call open; we can choose any collection of subsets we desire, as long as the above conditions are satisfied.

Given a topological space $(X, \mathcal{T})$, a subset A of X can be given a topology $\mathcal{T}_A$ by: $V \in \mathcal{T}_A$ iff $V = U \cap A$ for some $U \in \mathcal{T}$. This is called the **subspace topology** (also called the *relative topology*) on A. We say $(A, \mathcal{T}_A)$ is a (topological) **subspace** of $(X, \mathcal{T})$. The topology on A is **induced** by the topology on X.

A function from one topological space to another is **continuous** iff the preimage of every open set is open. A **homeomorphism** (denoted $\simeq$) is a bijection that is continuous and whose inverse is also continuous.

~~~~~~~~~

Connectedness

Informal Definition: A **topological invariant** is a property of a topological space that is preserved by homeomorphisms.

Example 1. We will soon prove that: If $A \simeq B$, then A is connected iff B is connected. In other words, "connectedness" is preserved by homeomorphisms, so it is a topological invariant.

Intuitively, we'd like to say $[0, 3]$ is connected, while $[0, 1] \cup [2, 3]$ is not.

Definition 1. A topological space X is **connected** iff it is *not* equal to the union of two disjoint nonempty open subsets.

Example 2. Let $X = (0, 1) \cup (2, 3) \subset \mathbb{R}$. (Note: X is implicitly assumed to inherit the subspace topology from $\mathbb{R}$.

Q: Is each of $(0, 1)$ and $(2, 3)$ open in X? [1]

Q: Is X connected? [2]

Example 3. Let $X = [0, 1) \cup (1, 2) \cup [5, 7] \subset \mathbb{R}$.

Q: Is each of $[0, 1)$, $(1, 2)$, and $[5, 7]$ open in X? [3]

Q: Is X connected? Can you write X as the union of *two* disjoint open subsets? [4]

Theorem 1. $\mathbb{R}$ is connected.

Proof. Suppose towards contradiction that $\mathbb{R}$ is not connected. Then, by definition, $\exists A, B \subset \mathbb{R}$ such that $\mathbb{R} = A \cup B$, where A and B are disjoint nonempty open subsets of $\mathbb{R}$. Pick arbitrary points $a \in A$ and $b \in B$. Without loss of generality, we can assume $a < b$. Let $A' = [a, b] \cap A$. Let $z = \text{lub}(A')$ (A' has a least upper bound because every bounded nonempty subset of $\mathbb{R}$ has a lub — this is called the Completeness Axiom for real numbers). Now, we claim that $z \notin A$ and $z \notin B$. Proof of claim: Extra Credit. But this gives us a contradiction, since $z \in \mathbb{R} = A \cup B$. □

[1] Yes. Why? This isn't as trivial as it seems; you need to think about subspace topology!

[2] No. Why?

[3] Yes. Why?

[4] Let $U = [0, 1)$, $V = (1, 2) \cup [5, 7]$; then U and V are disjoint, each is nonempty and open in X, and $X = U \cup V$.
~~~~~~~~~

Theorem 2. $A \subseteq \mathbb{R}$ is connected iff A is an interval (open, closed, or half open; infinite or half-infinite).

Proof: Extra Credit.

Theorem 3. The continuous image of a connected set is connected; i.e, if $f : X \to Y$ is a continuous map between topological spaces, and if X is connected, then f(X) is connected.

Proof: Homework.

Note. In the above theorem, $f(X)$ is guaranteed to be connected, but Y may or may not be connected.

Corollary 4. If X is connected, and Y is homeomorphic to X, then Y is connected (i.e., connectedness is a topological invariant).

Proof: Homework.

Example 4. Prove $[a, b) \not\simeq (c, d)$.

Sketch of Proof: Suppose towards contradiction that there exists a homeomorphism $h : [a, b) \to (c, d)$. Let $X = [a, b) - \{a\}$, $Y = (c, d) - \{h(a)\}$. It is easy to show that Y is not connected, but X is connected (since $X \simeq \mathbb{R}$). It is also easy to show that the restriction $h|_X : X \to Y$ is a homeomorphism, which implies that Y must be connected. This gives us the desired contradiction.

Q: How would you prove that $[a, b)$ is not homeomorphic to a circle (denoted $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$)?
Ans: we will prove this rigorously later; here is an informal proof: You need to remove at least two points from S^1 to make it disconnected, but you can disconnect $[a, b)$ by removing only one point.

Theorem 5. A topological space X is connected iff it contains no proper subset which is both open and closed in X.

Proof: Homework.

Theorem 6. If A and B are connected subspaces of a topological space X, and if $A \cap B \neq \phi$, then $A \cup B$ is connected.

Proof: Homework.