Quiz 4. Math 460, Topology. Name: ______________________

Instructor: Ramin Naimi Fri 31 Oct 2003

Closed book. Closed Notes. 25 points per problem. Please write very legibly.

Do **only two** of the following five problems. Extra Credit problems do not carry any points; so do not spend any time on them unless you're sure you've done your best with the problems that do carry points.

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1. (a) Let P be the quotient space obtained by identifying opposite points on the unit circle. Find a homeomorphism from P to S^1 (or vice versa). You don't have to prove that your map is a homeomorphism.

   (b) Find an equivalence relation $\sim$ on $I^2 = [0,1] \times [0,1]$ such that the quotient space $I^2/\sim$ is homeomorphic to a cylinder, $S^1 \times I$. Find a homeomorphism from $I^2/\sim$ to $S^1 \times I$ (or vice versa). You don't have to prove that your map is a homeomorphism.

2. State, without proof, whether or not each of the following topological spaces is a manifold (with or without boundary). If you claim that it is a manifold, give its dimension and boundary (without proof), and state whether or not it is compact (without proof). If you claim that it is not a manifold, give a *brief* reason (about ten words or less).

   (a) $\mathbb{R}^2_+ - \{(0,0)\}$ (b) $\mathbb{R}^2_+ - \{(x,y) \in \mathbb{R}^2 \mid 2 < x \leq 3,\ y = 0\}$ (c) $\mathbb{R}/\{x \sim (x+2)\}$

   (d) $[B_2(0,0) \cup B_2(6,0)]/\{\forall (x,y) \in \overline{B_1(0,0)},\ (x,y) \sim (x+6,y)\}$

   (e) $[B_2(0,0) \cup B_2(6,0)]/\{\forall (x,y) \in B_1(0,0),\ (x,y) \sim (x+6,y)\}$

3. Draw a series of pictures (stages of a deformation) to prove that a punctured torus (a torus with a small open disk removed) is isotopic to the surface drawn in part (a) below. Then give an argument to prove that a punctured torus is not homeomorphic to the surface drawn in part (b) below.

   (a) (b)

4. Prove the following theorem: S^1 cannot be embedded in $\mathbb{R}$.

5. Prove (without using Theorem 2 of Section 9) that $\mathbb{R}\mathrm{P}^2 \not\simeq S^2$. Hint: (i) *Jordan Curve Theorem*: Every embedded circle $C \subset S^2$ separates S^2. (ii) Show there exists a non-separating circle $C \subset \mathbb{R}\mathrm{P}^2$. (iii) Assume $\mathbb{R}\mathrm{P}^2 \simeq S^2$, and use (i) and (ii) to get a contradiction.

6. *Definition* A **component** of a topological space X (sometimes also called a *connected component*, for emphasis) is a connected subset of X that is not a subset of any other connected subset of X (in short, it's a maximal connected subset of X).

   Let X be a topological space. Prove that every component of X is open and closed. (Hint: Let C be a component of X. Prove C is the union of all open connected subsets of X that intersect C.)

7. (a) Give an example of a collection $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ of nested nonempty closed subsets of $\mathbb{R}$ whose intersection $\bigcap B_i$ is empty.

   (b) Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ be a collection of nested nonempty closed subsets of a compact topological space X. Prove their intersection $\bigcap B_i$ is nonempty.