The *long line* is an example of a connected, Hausdorff topological space that's locally homeomorphic to $\mathbb{R}$, but is not a manifold, because it is not second countable. Roughly speaking, it is $S_\Omega \times [0,1)$, where S_Ω is the smallest uncountable ordinal. To describe and understand it in detail, we first need some definitions. I am borrowing most of the following from Munkres's book, *Topology*, Second Edition, Prentice Hall.

Definition A relation R on a set X is an **order relation** if $\forall x, y \in X$, (1) xRy or yRx, (2) $\neg(xRx)$, and (3) R is transitive.

Definition A set X is **well ordered** by an order relation $<$ if every nonempty subset of X has a "smallest" element.

We construct the long line as follows. Let S_Ω be the smallest uncountable ordinal (if you don't know what that is, just think of it as an a well-ordered uncountable set). Let $L = S_\Omega \times [0,1) - \{(s_0, 0)\}$, where s_0 is the smallest element of S_Ω. Order L with the dictionary ordering, i.e., $(s,t) < (s',t')$ iff $s < s'$ or $[s = s'$ and $t < t']$.

We give L the **order topology**, which means a subset of L is open iff it is a union of open intervals (a,b) with $a, b \in L$, $a < b$. Then L with this topology is called the **long line**.

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Extra Credit Problems

1. Explain why we choose $L = S_\Omega \times [0,1)$ minus its smallest element instead of just letting $L = S_\Omega \times (0,1]$. Don't read the following hint until you've thought hard. Hint: The second way would not produce a connected space; why?

   Definition Let $\mathcal{T}$ be a topology on a set X. A **basis** for $\mathcal{T}$ is a subset $\beta \subseteq \mathcal{T}$ such that $\forall U \in \mathcal{T}$ and $\forall x \in U$, $\exists B \in \beta$ with $x \in B \subseteq U$.

2. (a) Prove that the long line is not **second countable**, i.e, it does not have a countable basis.
   (b) Prove that the long line cannot be embedded in $\mathbb{R}$, or in any $\mathbb{R}^n$. Hint: $\mathbb{R}^n$ has a countable basis.
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