Definition Let X be a topological space. A loop whose image is just one point in X is called a **trivial loop**. A loop is said to be **null-homotopic** iff it is homotopic as a loop to a trivial loop in X.

1. (a) Show that every loop in $\mathbb{R}^2$ is null-homotopic.

   (b) Show that any two loops in $\mathbb{R}^2$ are homotopic to each other as loops. Do this twice, once for each method below.

   Method 1: Use Part (a) above, and the fact that being homotopic as loops is an equivalence relation.

   Method 2: Construct a homotopy.

2. (a) Draw three loops on the torus, T^2, such that no two of them are homotopic to each other as loops. (No proof necessary). Can you find more than three?

   (b) How many loops are there on the annulus $S^1 \times I$ such that no two of them are homotopic to each other as loops? Support your answer by constructing the loops (but you don't need to prove that no two of them are homotopic to each other as loops).

3. Recall that $\mathbb{RP}^2$ is defined as (this is one of two definitions we have seen): the closed unit disk with **antipodal** (i.e., opposite) points on its boundary identified; $\mathbb{RP}^2 = D^2/\{\forall x \in \partial D^2,\ x \sim -x\}$. Let $q : D^2 \to \mathbb{RP}^2$ be the quotient map.

   (a) Let A be the horizontal diagonal in D^2, i.e., $A = \{(x, y) \in D^2 \mid y = 0\}$. Let $\alpha = q(A) \subset \mathbb{RP}^2$. Then α is a closed curve in $\mathbb{RP}^2$. Why? Technically, α is not really a loop. Why?

   (b) Give a homeomorphism h from I to A.

   (c) Explain why the composition $q \circ h$ is a loop in $\mathbb{RP}^2$. What is the image of this loop? Do you think this loop is null-homotopic (just Y or N, without proof)?

   (d) Define $g : I \to D^2$ by $g(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $q \circ g$ is a loop in $\mathbb{RP}^2$. Why? Prove that the loop $q \circ g$ is null-homotopic.

4. How many loops are there on $\mathbb{RP}^2$ such that no two of them are homotopic to each other as loops?

Extra Credit Problems

5. Prove that the Möbius band M is not orientable, by giving a map $h : D^2 \to M$ and an isotopy between h and its mirror image.

6. (a) Represent $3T^2$ as a polygon with edges identified appropriately.

   (b) Represent $3\mathbb{RP}^2$ as a polygon with edges identified appropriately.