1. State which of the following spaces are homeomorphic to each other. You do not need to prove your answers; but give brief explanations or draw pictures (or both) to support them.

   (a) $X = \overline{B_1(0,0)} \subset \mathbb{R}^2$ (i.e., X is the closed unit disk in the plane).

   (b) $Y = X - B_{0.5}(0,0)$. (c) Y° (the interior of Y, where Y is viewed as a subspace of $\mathbb{R}^2$).

   (d) $X/\partial X$. (This means identify the whole boundary of X into one point – recall that $\partial X = S^1$.)

   (e) $S^1 \times [0,1]$. (f) $S^1 \times (0,1)$. (g) $S^1 \times [0,1)$. (h) $S^1 \times \mathbb{R}$. (i) $\mathbb{R}^2$

   (j) $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$. (k) $S^2 - \{(1,0,0)\}$. (l) $S^2 - \{(1,0,0),(-1,0,0)\}$.

   (m) $S^2 - B_{0.5}(1,0,0)$.

2. Which of the following letters, when viewed as subsets of $\mathbb{R}^2$, are manifolds? Give a brief explanation for each one that is not a manifold.

   A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

3. How many non-homeomorphic 1-manifolds do you think there are? List as many as you can think of.

4. Is an open ball minus its center a manifold? More precisely, let $x \in \mathbb{R}^n$, $r > 0$. Is $B_r(x) - \{x\}$ a manifold? Prove your answer.

5. (a) Is $S^1/\{(1,0) \sim (-1,0)\}$ a manifold? Explain.

   (b) Is $S^1/\{(x,y) \sim (-x,-y)\}$ a manifold? Explain.

6. Give an example of topological spaces X and Y such that X is locally homeomorphic to Y but Y is not locally homeomorphic to X.

Extra Credit Problems

7. *Definition* A **component** of a topological space X (sometimes also called a *connected component*, for emphasis) is a connected subset of X that is not a subset of any other connected subset of X (in short, it's a maximal connected subset of X).

   Let X be a topological space. Prove that every component of X is open and closed. (Hint: Let C be a component of X. Prove C is the union of all open connected subsets of X that intersect C.)

8. *Definition* A topological space X is path connected if $\forall x, y \in X$ there is a **path** from x to y; i.e., there is a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

   Prove whether each of the following is true or false. (You may give a sketch of the proof, containing all the necessary ingredients, but perhaps missing some of the tedious details.)

   (a) Every connected space is path connected.

   (b) Every path connected space is connected.

   Hint: For one of the above, consider the following subset $X \subset \mathbb{R}^2$, with the subspace topology inherited from $\mathbb{R}^2$, called the *topologist's sine curve.*

   $$X = \{(x,y) \in \mathbb{R}^2 \mid (x = 0 \text{ and } -1 \le y \le 1) \text{ or } (x > 0 \text{ and } y = \sin(1/x))\}$$