1. (a) Let $X_1 = \mathbb{R}$, $\mathcal{T}_1 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$. Prove that $\mathcal{T}_1$ is a topology.

   (b) Let $(X_2, \mathcal{T}_2)$ be $\mathbb{R}$ with the standard topology (i.e., the topology induced by the Euclidean metric). With $(X_1, \mathcal{T}_1)$ given above, let $f : X_1 \to X_2$ be given by $f(x) = x$. Is f continuous? Is f^{-1} continuous? Prove your answers.

2. For $i = 1, 2$, let $(X_i, \mathcal{T}_i)$ be a topological space.

   (a) Show that if $\mathcal{T}_1$ is the discrete topology, then every function $f : X_1 \to X_2$ is continuous.

   (b) Show that if $\mathcal{T}_2$ is the indiscrete topology, then every function $f : X_1 \to X_2$ is continuous (regardless of what $\mathcal{T}_1$ is).

3. For $i = 1, 2, 3$, let $(X_i, \mathcal{T}_i)$ be a topological space. Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous maps. Prove that their composition $g \circ f : X_1 \to X_3$ is continuous.

Extra Credit Problems

4. Let X and Y be topological spaces. Prove $f : X \to Y$ is continuous iff it takes limit points to limit points; i.e., f is continuous iff for every $p \in X$ that's a limit point of a subset $A \subseteq X$, $f(p)$ is a limit point of $f(A)$.

5. *Definition* Let (X, d) be a metric space. We say a sequence of points $a_1, a_2, a_3, \dots \in X$ **converges** to a point $p \in X$ if $\forall \epsilon > 0$ $\exists M$ such that $\forall n > M$, $d(a_n, p) < \epsilon$. We write $\lim_{n\to\infty} a_n = p$, and say the sequence $a_1, a_2, a_3, \cdots$ is **convergent**.

   Let (X_1, d_1) and (X_2, d_2) be metric spaces. Prove that $f : X_1 \to X_2$ is continuous iff for every convergent sequence $a_1, a_2, a_3, \dots \in X_1$, $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.